EXERCISES ON UNITARY DIAGONALIZATION
AND POSITIVE DEFINITE MATRICES

Ex. 1 Orthogonally diagonalize the following matrices:

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 3 \\
1 & 3 & 1 \\
3 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}.
\]

I claim that you can diagonalize the first and third matrices without pencil and paper. You can also find an eigenvalue of the second by inspection.

Ex. 2 Suppose \(B\) is a real, symmetric \(3 \times 3\) matrix such that \((1, 0, 1)^T \in \text{Null}(B - I_3)\), and \((1, 1, -1)^T \in \text{Null}(B - 2I_3)\). If \(\det(B) = 12\), find \(B\).

Ex. 3 Let \(W\) be a hyperplane in \(\mathbb{R}^n\), and let \(H\) be the reflection through \(W\).

(a) Express \(H\) in terms of \(P_W\) and \(P_W^\perp\).

(b) Show that \(P_WP_W^\perp = P_W^\perp P_W\).

(c) Simultaneously orthogonally diagonalize \(P_W\) and \(P_W^\perp\).

Ex. 4 Prove that a real symmetric matrix \(A\) whose only eigenvalues are \(\pm 1\) is orthogonal.

Ex. 3 Suppose \(A \in \mathbb{R}^{n \times n}\) is symmetric. Show the following:

(i) \(\mathcal{N}(A) \perp = \text{im}(A)\).

(ii) \(\text{im}(A) \perp = \mathcal{N}(A)\).

(iii) \(\text{col}(A) \cap \mathcal{N}(A) = \{0\}\).

(iv) Conclude from (iii) that if \(A^k = O\) for some \(k > 0\), then \(A = O\).

Ex. 5 Let \(S\) be a skew Hermitian \(n \times n\) matrix. Show the following:

(i) For all \(v \in \mathbb{C}^n\), \(v^H Av\) is pure imaginary.

(ii) All eigenvalues of \(S\) are pure imaginary. (Hint: use (i).)

(iii) If \(n\) is odd, then \(\det(S)\) is pure imaginary, and if \(n\) is even, then \(\det(S)\) is real.
(iv) If $S$ is skew symmetric, then $\det(S) = 0$ if $n$ is odd, and $\det(S) \geq 0$ if $n$ is even.

**Ex. 6** Which of the following statements are always true, which are sometimes true and which are always false? Discuss your reasoning.

(i) For any square matrices $A$ and $B$ over $\mathbb{R}$, $e^{A+B} = e^A e^B$.

(ii) Every unitary matrix is diagonalizable.

(iii) The eigenvalues of a unitary matrix are real.

(iv) If $A$ is skew symmetric, $e^A$ is orthogonal.

(iv) If $A$ is skew Hermitian, $e^A$ is unitary.

(v) If the determinant of $A$ is nonzero, then $A^T A$ is positive definite.

**Ex. 7** For the following pairs $A, B$ of symmetric matrices, determine whether $A$ and $B$ are congruent or not.

(i) $A$ and $B$ have the same characteristic polynomial and distinct eigenvalues.

(ii) $\det(A) < 0$, $\det(B) > 0$.

(iii) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

**Ex. 8** Show that if $A \in \mathbb{R}^{n \times n}$ is positive definite, then every diagonal entry of $A$ is positive. Also show that $rA$ is positive definite if $r > 0$ and negative definite if $r < 0$.

**Ex. 9** When is $e^A$ positive definite? Can $e^A$ ever be negative definite or indefinite?