Math 422
Practice Midterm Exam

The midterm will have 4 or 5 problems, not counting parts!

1. Let G be a finite group acting on a set X. We say that $x \in X$ is a G-fixed point if the stabilizer G_x of x is all of G.

(i) Show that $Z = \{g \in G \mid g \cdot x = x \ \forall \ x \in X\}$ is a normal subgroup of G.

(ii) Show that if G acts transitively on X, then G has no fixed points.

(iii) Suppose G acts transitively on X. Show that $o(G) \geq |X|$.

(iv) Suppose G is simple and acts transitively on X. Show that if $|X| = n$, then G is isomorphic to a subgroup of S_n.

(v) Suppose $X = G$ and the action is by conjugation. Show that the number of fixed points of G divides the order of G.

(vi) True or False: The number of fixed points always divides the order of G.

2. Let a, b, c be odd integers. Decide whether or not the polynomial $x^4 + ax^3 + bx^2 + cx + 1$ is irreducible (over $\mathbb{Z}$).

3. Let $\mathbb{F}$ be a field and $f(x) \in \mathbb{F}[x]$ a polynomial of degree m.

(i) What is the dimension over $\mathbb{F}$ of the quotient ring $K = \mathbb{F}[x]/(f(x))$? Prove your answer.

(ii) When is K a field?

(iii) If K is the splitting field of a polynomial over $\mathbb{F}$, what is the order of the Galois group $G(K, \mathbb{F})$?

4. Describe all groups of order 9.

5. Let p and q be distinct primes.

(i) Every group of order pq is abelian. Justify your answer.

(ii) Let $p = 5$ and $q = 13$, and let G have order 65. How many elements of orders 5 and 13 respectively does G contain?

(iii) True or False: If $o(G) = 65$, then G is abelian? Justify your answer.

6. Let K be the field obtain by adjoining $2^{1/2} + 3^{1/3}$ to the rationals $\mathbb{Q}$. What is the order of $G(K, \mathbb{Q})$? Is K normal over $\mathbb{Q}$?