MATH 422-501
Assignment 3

Due Wednesday November 2, 2005
(posted Sunday, Oct. 23)

Note: All rings contain an identity.

Exercise 1 Give an example of a commutative ring for which unique factorization fails.

Exercise 2 Let V be a finite dimensional vector space. Show that if U and W are subspaces of V, then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

This is called the Hausdorff Intersection Formula.

Exercise 3 Let V be a finite dimensional vector space, and let $W_1, \ldots, W_k$ be subspaces of V such that $V = W_1 + \cdots + W_k$. Show that $V = W_1 \oplus \cdots \oplus W_k$ if and only if $\dim(V) = \sum_{i=1}^{k} \dim(W_i)$.

Exercise 4 Find an example of a finitely generated module over a ring R (with identity) having a submodule N that is not finitely generated.

Exercise 5 Let $\mathbb{F}$ be a field and let S be a finite subset of $\mathbb{F}$ which is closed under addition and multiplication.

(i) Show that S is a subfield of $\mathbb{F}$.

(ii) Show that if the characteristic of $\mathbb{F}$ is $p > 0$, then $\mathbb{F}' = \{m1 \mid 0 \leq m < p\}$ is a subfield of $\mathbb{F}$ isomorphic to $\mathbb{F}_p$.

(iii) Show that if $\mathbb{F}$ is a Galois field of characteristic p, then $|\mathbb{F}| = p^n$ for some $n > 0$, and describe what n is.

(iv) If $\mathbb{F}$ is as in part (iii) and $q = p^n$, show that $a^q = a$ for all $a \in \mathbb{F}$.

Exercise 6 Show that if $\mathbb{F}$ is a Galois field of characteristic p, then every element of $a \in \mathbb{F}$ is a p-th power: that is, $a = b^p$ for some $b \in \mathbb{F}$.

Exercise 7 Let $\mathbb{F}$ be a field and suppose $g(x)$ and $h(x)$ are irreducible polynomials over $\mathbb{F}$. Show that if g and h are not associates, then they cannot have a common root in any extension field of $\mathbb{F}$.

Exercise 8 Let $\mathbb{F}$ be a subfield of a field $\mathbb{K}$. Show that if $a, b \in \mathbb{K}$ are algebraic over $\mathbb{F}$ of degrees m and n, where $(m, n) = 1$, then $[\mathbb{F}(a, b) : \mathbb{F}] = mn$.

Exercise 9 Problems 2 and 9 on page 236 of Herstein.