Exercise 1 Suppose $G$ is a group of order 30.
(1) Show that $G$ has a normal subgroup of order 15.
(2) Determine the largest possible number of non-isomorphic $G$. (Note: the answer is 4.)

Exercise 2 Let $G$ be the dihedral group of order 18. That is, $G = \langle a, b \mid a^9 = b^2 = e, bab^{-1} = a^{-1} \rangle$. Let $H = \langle a^3 \rangle$ and $K = \langle b \rangle$. Find a representative for each double coset $HxK$ of $H$ and $K$ and also compute $|HxK|$.

Exercise 3 Find all finite abelian groups of orders 64, $11^6$ and $2^43^4$.

Exercise 4 Let $G$ be a finite abelian group. Let $\hat G$ denote the set of all homomorphisms $\phi : G \to \mathbb{C}^\ast$.
(1) Show that $(\hat G)$ is a finite abelian group. In fact, show that $o(\hat G) = o(G)$.
(2) Show that $G \cong \hat G$.
(3) Show that if $\phi \neq 1$, then $\sum_{g \in G} \phi(g) = 0$.

Exercise 5 For each $\sigma \in S_n$, let
$$\text{sgn}(\sigma) := \prod_{i<j} \frac{\sigma(i) - \sigma(j)}{i-j}.$$ 

(i) Show that sgn defines a homomorphism $\text{sgn}: S_n \to \{\pm 1\}$.
(ii) Show that if $s$ is a transposition, then $\text{sgn}(\sigma) = -1$.

Exercise 6 Let $A_n \subset S_n$ denote the alternating group consisting of even permutations in $S_n$. Show that $A_n$ is simple if $n \geq 5$. Also, show that $A_4$ is not simple.

Exercise 7 Let $R$ be a ring in which every element satisfies $x^2 = x$. Show that $R$ is commutative.

Exercise 8 Let $F$ be a field and $\varphi : F \to R$ be a ring homomorphism. Show that $\varphi$ is either injective or the zero map.

Exercise 9 Does the ring $\mathbb{Z}^{2\times2}$ of all integer $2 \times 2$ matrices contain any nontrivial ideals? How about $\mathbb{Z}^{n\times n}$?

Exercise 10 Show that the map $\varphi : \mathbb{Z}_4 \to \mathbb{Z}_{10}$ defined by $\varphi(x) = 5x$ is a well defined homomorphism and find its kernel and image.