MATH 422-501
Assignment 2
Due Wednesday, Oct. 12, 2005

Exercise 1 Suppose G is a group of order 30.

(1) Show that G has a normal subgroup of order 15.

(2) Determine the largest possible number of non-isomorphic G. (Note: the answer is 4.)

Exercise 2 Let G be the dihedral group of order 18. That is, $G =< a, b \mid a^9 = b^2 = e,\ bab^{-1} = a^{-1} >$. Let $H =< a^3 >$ and $K =< b >$. Find a representative for each double coset HxK of H and K and also compute $|HxK|$.

Exercise 3 Find all finite abelian groups of orders 64, 11^6 and $2^4 3^4$.

Exercise 4 Let G be a finite abelian group. Let $\hat{G}$ denote the set of all homomorphisms $\phi : G \to \mathbb{C}^*$.

(1) Show that $\hat{(G)}$ is a finite abelian group. In fact, show that $o(\hat{G}) = o(G)$.

(2) Show that $G \cong \hat{G}$.

(3) Show that if $\phi \neq 1$, then $\sum_{g \in G} \phi(g) = 0$.

Exercise 5 For each $\sigma \in S_n$, let

$$\mathrm{sgn}(\sigma) := \prod_{i<j} \frac{\sigma(i) - \sigma(j)}{i - j}.$$

(i) Show that sgn defines a homomorphism sgn: $S_n \to \{\pm 1\}$.

(ii) Show that if s is a transposition, then $\mathrm{sgn}(\sigma) = -1$.

Exercise 6 Let $A_n \subset S_n$ denote the alternating group consisting of even permutations in S_n. Show that A_n is simple if $n \geq 5$. Also, show that A_4 is not simple.

Exercise 7 Let R be a ring in which every element satisfies $x^2 = x$. Show that R is commutative.

Exercise 8 Let $\mathbb{F}$ be a field and $\varphi : \mathbb{F} \to R$ be a ring homomorphism. Show that φ is either injective or the zero map.

Exercise 9 Does the ring $\mathbb{Z}^{2\times 2}$ of all integer 2×2 matrices contain any nontrivial ideals? How about $\mathbb{Z}^{n\times n}$?

Exercise 10 Show that the map $\varphi : \mathbb{Z}_4 \to \mathbb{Z}_{10}$ defined by $\varphi(x) = 5x$ is a well defined homomorphism and find its kernel and image.