

MATH 422-501
Assignment 2
Due Wednesday, Oct. 12, 2005

Exercise 1 Suppose G is a group of order 30.

- (1) Show that G has a normal subgroup of order 15.
- (2) Determine the largest possible number of non-isomorphic G . (Note: the answer is 4.)

Exercise 2 Let G be the dihedral group of order 18. That is, $G = \langle a, b \mid a^9 = b^2 = e, bab^{-1} = a^{-1} \rangle$. Let $H = \langle a^3 \rangle$ and $K = \langle b \rangle$. Find a representative for each double coset HxK of H and K and also compute $|HxK|$.

Exercise 3 Find all finite abelian groups of orders 64 , 11^6 and $2^4 3^4$.

Exercise 4 Let G be a finite abelian group. Let \hat{G} denote the set of all homomorphisms $\phi : G \rightarrow \mathbb{C}^*$.

- (1) Show that (\hat{G}) is a finite abelian group. In fact, show that $o(\hat{G}) = o(G)$.
- (2) Show that $G \cong \hat{\hat{G}}$.
- (3) Show that if $\phi \neq 1$, then $\sum_{g \in G} \phi(g) = 0$.

Exercise 5 For each $\sigma \in S_n$, let

$$\text{sgn}(\sigma) := \prod_{i < j} \frac{\sigma(i) - \sigma(j)}{i - j}.$$

- (i) Show that sgn defines a homomorphism $\text{sgn} : S_n \rightarrow \{\pm 1\}$.
- (ii) Show that if s is a transposition, then $\text{sgn}(\sigma) = -1$.

Exercise 6 Let $A_n \subset S_n$ denote the alternating group consisting of even permutations in S_n . Show that A_n is simple if $n \geq 5$. Also, show that A_4 is not simple.

Exercise 7 Let R be a ring in which every element satisfies $x^2 = x$. Show that R is commutative.

Exercise 8 Let \mathbb{F} be a field and $\varphi : \mathbb{F} \rightarrow R$ be a ring homomorphism. Show that φ is either injective or the zero map.

Exercise 9 Does the ring $\mathbb{Z}^{2 \times 2}$ of all integer 2×2 matrices contain any nontrivial ideals? How about $\mathbb{Z}^{n \times n}$?

Exercise 10 Show that the map $\varphi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_{10}$ defined by $\varphi(x) = 5x$ is a well defined homomorphism and find its kernel and image.