Practice Exam. Math 257-316(Bui)

I. Solve the boundary-value problem

$$\begin{aligned} x^2u_{xx} + xu_x + u_{yy} &= 0,\ 1 < x < e, 0 < y < 1 \\ u(x,0) &= 0,\ u(e,y) = 0 \\ u(x,0) &= f(x),\ u(x,1) = 0. \end{aligned}$$

Solution 1) Since the source is zero and the boundary conditions on two parallel sides are zero, we may use the method of separation of variables and set $u = XY$. We get

$$x^2X' + xX' = -\lambda^2 X;\ X(1) = 0 = X(e).$$

The indicial equation is $r(r-1) + r + \lambda^2 = 0$ and the roots are $r = \pm i\lambda$. Hence

$X(x) = c_1\cos(\lambda \ln x) + c_2 \sin(\lambda \ln x)$; $X(1) = 0$ gives $c_1 = 0$, $X(e) = 0$ gives $\lambda = n\pi$.

We now have $Y' + n^2\pi^2 Y = 0$, and thus,

$$u(x,y) = \sum_{n=1}^{\infty}\{a_n\cosh(n\pi y) + b_n\sinh(n\pi y)\}\sin(n\pi \ln x)$$

with

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi \ln x).$$

Note: To use the sine Fourier series expansion in order to get a_n, the right hand side of the above equation should have only terms of the form $\sin(n\pi x)$. **We should make a change of variables before using the Fourier series formula.**

Make the change of variable $\ln x = t$, i.e. $x = e^t$ then $x = 1$, $x = e$ correspond to $t = 0, t = 1$ respectively. We now have

$$f(e^t) = \sum_{n=1}^{\infty} a_n \sin(n\pi t)$$

and thus,

$$a_n = 2\int_0^1 f(e^t)\sin(n\pi t)dt = 2\int_1^e f(x)\sin(n\pi \ln x)x^{-1}dx.$$

An easy calculation gives

$$b_n = -a_n\tanh(n\pi).$$

2 A second way of doing it is as follows. Re write the equation as

$$(xu_x)_x + \frac{1}{x}u_{yy} = 0$$

and consider the self adjoint problem

$$(xX')' = -\lambda^2\frac{1}{x}X;\ X(1) = 0 = X(e).$$

The weight function is $r(x) = 1/x$.

The eigenvalues are $n^2\pi^2$ and the normalized eigenfunctions are

$$\varphi_n(x) = \frac{1}{\sqrt{\alpha_n}} X_n = \frac{1}{\sqrt{\alpha_n}} \sin(n\pi \ln x); \; \alpha_n = \int_1^e \sin^2(n\pi \ln x)\frac{1}{x}dx.$$

Set

$$u(x, y) = \sum_{n=1}^{\infty} a_n(y)\varphi_n(x).$$

We have

$$\sum_{n=1}^{\infty}\{-\frac{1}{x}n^2\pi^2 a_n(y) + \frac{1}{x}a_n''(y)\}\varphi_n(x) = 0.$$

Thus

$$a_n''(y) - n^2\pi^2 a_n(y) = 0; \; n = 1, ..$$

We get

$$u(x, y) = \sum_{n=1}^{\infty}\{A_n \cosh(n\pi y) + B_n \sinh(n\pi y)\}\varphi_n(x).$$

Now the condition $u(x, 0) = f(x)$ gives

$$A_n = \int_1^e f(x)\varphi_n(x)\frac{1}{x}dx = \int_1^e f(x)\varphi_n(x)r(x)dx$$

where $r(x)$ is the weight function. The condition $u(x, 1) = 0$ yields $B_n = -\tanh(n\pi)A_n$.

II. Solve the initial boundary-value problem

$$\begin{aligned} u_t &= u_{xx} - u + \sin x, \; 0 < x < 1, 0 < t \\ u(0, t) &= 0, \; u(1, t) + 2u_x(1, t) = 0 \\ u(x, 0) &= f(x). \end{aligned}$$

Solution. Since the source depends only on the space variable, we may write u as the sum of two functions, one depending only on x which represents the steady part due to the source and one depending on x, t without the source. Set $u(x, t) = v(x) + w(x, t)$ with

$$0 = v' - v + \sin x; \; v(0) = 0 = v(1) + 2v'(1).$$

It is clear that

$$v(x) = c_1 \cosh x + c_c \sinh x + v_{particular}$$

The method of unundertmined coefficients with $v_p = A \sin x + B \cos x$ gives $v_p = \frac{1}{2} \sin x$.

Using the boundary conditions we get c_1, c_2.

2) For w we have the problem

$$\begin{aligned} w_{tt} &= (w_x)_x - w + 0 \\ w(0, t) &= 0, \; w(1, t) + 2w_x(1, t) = 0 \\ w(x, 0) &= u(x, 0) - v(x) = f(x) - v(x) \end{aligned}$$

We can now use the method of separation of variables with $w = XT$ We get

$$(X')' = -\lambda^2 X;\ X(0) = 0 = X(1) + 2X'(1).$$

We have

$$X_n(x) = \sin(\lambda_n x);\ \lambda_n \text{ are the intersection of } y = \tan(\lambda) \text{ with } y = -2\lambda.$$

The weight function is 1 and the normalized eigenfunctions are

$$\varphi_n(x) = \frac{1}{\sqrt{\alpha_n}} X_n;\ \alpha_n = \int_0^1 X_n^2(x).1dx.$$

We have

$$T'/T = -\lambda_n^2 - 1;\ T_n(t) = T_n(0)e^{-(\lambda_n^2+1)t}.$$

Thus,

$$w(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n^2+1)t}\varphi_n(x).$$

With $w(x,0) = f(x) - v(x)$, we get

$$a_n = \int_0^1 \{f(x) - v(x)\}\varphi_n(x).1dx.$$

III. Solve the initial boundary-value problem

$$\begin{aligned} u_{tt} &= u_{xx} + t\cos(\frac{3\pi}{2}x);\ ) < x < 1, 0 < t \\ u_x(0,t) &= 0 = u(1,t) \\ u(x,0) &= 0 = u_t(x,0). \end{aligned}$$

Solution. Note that $u_{xx} = (u_x)_x$ and thus our weight function is $r(x) = 1$. Consider the self adjoint problem

$$(X')' = -\lambda^2 1X;\ X'(0) = 0 = X(1).$$

It is easy to check that

$$\lambda = (2n+1)\pi/2;\ X_n(x) = \cos(\frac{2n+1}{2}\pi x).$$

The normalized eigenfunctions are

$$\varphi_n(x) = \sqrt{2}\cos(\frac{(2n+1)\pi x}{2});\ n = 0, ..$$

Set

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\varphi_n(x)$$

and we have

$$\sum_{n=0}^{\infty}\{a_n''(t) + (2n+1)^2\frac{\pi^2}{4}a_n(t)\}\varphi_n(x) = t\cos(\frac{3\pi x}{2}).$$

Thus,

$$a_n''(t) + (2n+1)^2\frac{\pi^2}{4}a_n(t) = 0;\ n \neq 1$$

and

$$\{a_1''(t) + \frac{9\pi^2}{4}a_1(t)\}\varphi_1(x) = t\cos(\frac{3\pi x}{2}).$$

Hence

$$a_1''(t) + \frac{9\pi^2}{4}a_1(t) = \frac{t}{\sqrt{2}}.$$

The initial conditions give $a_n(0) = 0 = a_n'(0)$ for all n.

Solving

$$a_n''(t) + \frac{(2n+1)\pi^2}{4}a_n(t) = 0;\ a_n(0) = 0 = a_n'(0);\ \ n \neq 1$$

gives $a_n(t) = 0\ n \neq 1$.

We now consider

$$a_1''(t) + \frac{9\pi^2}{4}a_1(t) = \frac{t}{\sqrt{2}};\ a_1 = 0 = a'(0).$$

We have

$$a_1(t) = A\cos(\frac{3\pi t}{2}) + B\sin(\frac{3\pi t}{2}) + a_1^{part}$$

The particular solution is of the form $p_1(t) = c_1t + c_2$. We get by an easy calculation

$$a_1(t) = \frac{2\sqrt{2}}{9\pi^2}\{-\cos(\frac{3\pi t}{2}) + 1\}$$

and $u(x,t) = a_1(t)\varphi_n(x)$.