Take te^{-x} and expand it in terms of φ_n,

$$te^{-x} = t\sum_{n=1}^{\infty} d_n \varphi_n(x) \quad ; \quad d_n = \int_0^1 e^{-x}\varphi_n(x)\underbrace{e^{-x}}_{\text{weight fun.}}dx$$

$$d_n = \int_0^1 e^{-2x}\varphi_n(x)\,dx \,.$$

Then

$$\cancel{e^{-x}}\sum_{n=1}^{\infty} c_n'(t)\varphi_n(x) = \cancel{e^{-x}}\sum_{n=1}^{\infty} -\frac{4n^2\pi^2+1}{4}c_n(t)\varphi_n(x) + t\cancel{e^{-x}}\sum_{n=1}^{\infty} d_n\varphi_n(x)$$

$$\Rightarrow \quad c_n'(t) + \frac{4n^2\pi^2+1}{4}c_n(t) = td_n \quad ; \quad n = 1, \ldots$$

$$\Rightarrow \quad \left(c_n(t)\,e^{\frac{4n^2\pi^2+1}{4}t}\right)' = d_n\, t e^{\frac{4n^2\pi^2+1}{4}t}$$

$$\Rightarrow \quad c_n(t) = c_n(0)e^{-\frac{4n^2\pi^2+1}{4}t} + e^{-\frac{4n^2\pi^2+1}{4}t} d_n \int_0^t s e^{\frac{4n^2\pi^2+1}{4}s}\,ds \,.$$

(iv) Finally to get $c_n(0)$, we use the initial cdt.

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n(0)\varphi_n(x) \Rightarrow c_n(0) = \int_0^1 f(x)\varphi_n(x)\underbrace{e^{-x}}_{\text{weight fun.}}dx$$