Solution. With force depending on both x and t, we use the method of eigenfunction expansions.

(i) $e^{-x}u_t = (e^{-x}u_x)_x + te^{-x}\cdot e^{-x}$. $\quad (X'' - X') = 0$

$r(x) = e^{-x}$ $\quad (e^{-x}X')' = 0 \Rightarrow$

(ii) Now we consider:

$$(e^{-x}X')' = -\lambda X e^{-x}; \quad X(0) = 0 = X(1)$$

$$\Rightarrow X'' - X' + \lambda X = 0 \Rightarrow r^2 - r + \lambda = 0 \Rightarrow r = \frac{1 \pm \sqrt{1-4\lambda}}{2}$$

$$X(x) = e^{\frac{x}{2}}\left\{c_1 \cos \frac{\sqrt{4\lambda-1}}{2}x + c_2 \sin \frac{\sqrt{4\lambda-1}}{2}x\right\}.$$

$X(0) = 0 = c_1 \Rightarrow X(1) = 0 \Rightarrow \sqrt{4\lambda-1} = 2n\pi$. $\quad \boxed{\lambda_n = \frac{4n^2\pi^2+1}{4}}$

$$X_n = e^{\frac{x}{2}}\sin(n\pi x)$$

$$\alpha_n = \int_0^1 e^{x}\sin^2(n\pi x)\, e^{-x}\,dx = \int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}.$$

Normalized eigenfunctions $\varphi_n(x) = \sqrt{2}\, e^{\frac{x}{2}}\sin(n\pi x)$.

(iii) Solution of the form $u(x,t) = \sum_{n\geq 1}^{\infty} c_n(t)\varphi_n(x)$. So:

$$\underbrace{e^{-x}\sum_{n\geq 1}^{\infty} c_n'(t)\varphi_n(x)}_{e^{-x}u_t} = \underbrace{\sum_{n\geq 1}^{\infty} -\frac{4n^2\pi^2+1}{4}c_n(t)\varphi_n(x)e^{-x}}_{(e^{-x}u_x)_x} + te^{-x}\cdot e^{-x}.$$