(ii) $Y'' - Y = \lambda_n^2 Y \Rightarrow Y'' - (1+\lambda_n^2) Y = 0$

$$Y_n = c_1 e^{\sqrt{1+\lambda_n^2}\, y} + c_2 e^{-\sqrt{1+\lambda_n^2}\, y}.$$

Y_n bounded as $y \to +\infty \Rightarrow c_1 = 0$.

(iii) $$u(x,y) = \sum_{n=1}^{\infty} c_n e^{-\sqrt{1+\lambda_n^2}\, y} \varphi_n(x).$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x) \Rightarrow c_n = \int_0^1 f(x)\varphi_n(x)\,dx \quad ; n = 1 \ldots$$

II/ Solve the initial boundary-value problem

$$\begin{cases} u_{tt} = u_{xx} + \sin x \ ; & 0<x<1, \quad t>0 \\ u(0,t) = 0 = u(1,t) \\ u(x,0) = x(1-x); \quad u_t(x,0) = 0. \end{cases}$$

Sol. With the force depending only on 1 variable, it is easier to set

$$u(x,t) = \boxed{v(x)} + w(x,t)$$

instead of using the method of eigenfunction expansions.

(i) For $v(x) \Rightarrow v_{tt} = v_{xx} + \sin x = 0 \Rightarrow v'' = -\sin x$

$v(0) = 0 = v(1)$

$v(x) = c_1 + c_2 x + v_{part}$; $\quad v_{part} = A\sin x + B\cos x$

$v'_p = A\cos x - B\sin x \quad ; \ v''_p = -A\sin x - B\cos x$

$\Rightarrow A = 1; B = 0, \quad v = c_1 + c_2 x + \sin x$

$v(0) = 0 \Rightarrow c_1 = 0 \qquad v(1) = 0 \Rightarrow c_2 + \sin 1 = 0$

$v(x) = -(\sin 1)x + \sin x.$

(ii) Equation for $w(x,t)$.