Math257/316. Section 102 - Practice Test #2.

1/. Solve the boundary-value problem

$$\begin{cases} u_{xx}+u_{yy}-u=0, & 0<x<1,\ 0<y<1 \\ u(0,y)=0=u(1,y)+u_x(1,y) & \\ u(x,0)=f(x) \ ; & u(x,y) \text{ bounded as } y \to +\infty . \end{cases}$$

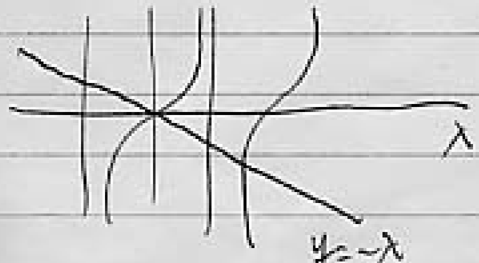
Solution. With RHS $=0$ and boundary conditions $=0$, we use the method of separation of variables. Set $u = X(x)Y(y)$, then:

$$\frac{Y''}{Y}-1+\frac{X''}{X}=0, \qquad X(0)=0=X(1)+X'(1).$$

(i) $X''=-\lambda^2 X$; $\quad X(0)=0=X(1)+X'(1)$

$X(x)=c_1\cos\lambda x + c_2 \sin\lambda x \quad ; X(0)=0 \Rightarrow c_1=0.$

$X(1)+X'(1)=0 \Rightarrow \sin\lambda+\lambda\cos\lambda=0 \Rightarrow \tan\lambda=-\lambda.$



λ_n = intersection of $y=\tan\lambda$ with line $y=-\lambda$.

$X_n = \sin(\lambda_n x)$; $\quad \alpha_n=\int_0^1 \sin^2(\lambda_n x)\,dx$

$$\alpha_n=\int_0^1 \frac{1-\cos(2\lambda_n x)}{2}\,dx = \frac{1}{2}-\frac{1}{4\lambda_n}\sin(2\lambda_n x)\Big|_0^1$$

$$=\frac{1}{2}-\frac{1}{4\lambda_n}\sin 2\lambda_n = \frac{1}{2}-\frac{1}{2\lambda_n}\sin\lambda_n\cos\lambda_n$$

$$=\frac{1}{2}\left(1-\frac{1}{\lambda_n}\cos\lambda_n\left[-\lambda_n\cos\lambda_n\right]\right)=\frac{1}{2}\left\{1+\cos^2\lambda_n\right\}.$$

Normalized eigenfunctions $\varphi_n(x)=\dfrac{\sin(n\pi x)}{\sqrt{\alpha_n}}$.