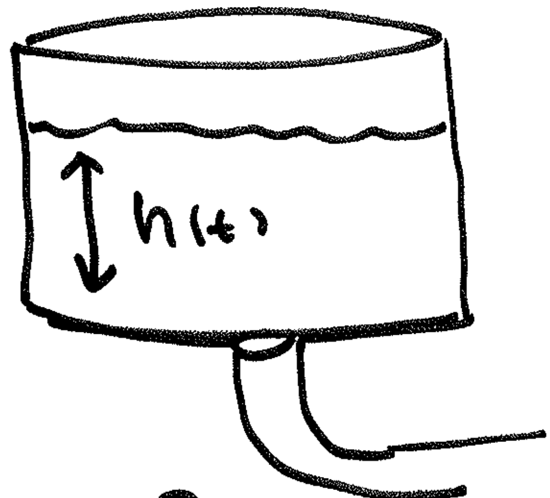
Math 101
April 12, 2010

Toricelli's Law



Tank filled with water to height $h(t)$ at time t.
Volume is $V(t)$.
How fast does it drain?

$$\frac{dV}{dt} = -a\sqrt{2gh}$$

where a = area of opening at bottom.

Problem (From HW)

Suppose tank is cylindrical with height 6 ft, radius 2 ft and hole is cylindrical with radius 1 inch. Show that h satisfies

$$\frac{dh}{dt} = \frac{-\sqrt{h}}{72}.$$

Sol $V(t) = \pi \cdot 2^2 \cdot h(t) = 4\pi h(t)$.

$$a = \pi \cdot \left(\frac{1}{12}\right)^2 = \frac{\pi}{144}$$

$$\frac{dV}{dt} = 4\pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{dV}{dt}\frac{1}{4\pi}$$

$$\Rightarrow \frac{dh}{dt} = \frac{-1}{4\pi}\frac{\pi}{144}\sqrt{2gh}.$$

Now $g = 32 \frac{ft}{sec^2} \Rightarrow \sqrt{2g} = \sqrt{64} = 8.$ (3)

$$\Rightarrow \frac{dh}{dt} = \frac{-1}{4 \cdot 144} \cdot 8\sqrt{h}$$

$$= \boxed{\frac{-1}{72}\sqrt{h}}.$$

Lets solve eq.

$$\boxed{\frac{dh}{dt} = k\sqrt{h}.}$$

where k is constant.

- It's clearly seperable.

$$\frac{dh}{dt} = k\sqrt{h} \Rightarrow \frac{dh}{\sqrt{h}} = k\,dt$$

$$\Rightarrow 2h^{1/2} = kt + C$$

$$\Rightarrow h^{1/2} = \frac{kt + C}{2}$$

$$\Rightarrow \boxed{h = \left(\frac{kt + C}{4}\right)^2 = \frac{k^2}{4}t + \frac{kCt}{2} + \frac{C^2}{4}.}$$

<u>Problem</u> 2(a) Suppose hole is drilled in side of cylindrical bottle and height h from 10 cm to 3 cm in 68 seconds. Using

$$\frac{dh}{dt} = k\sqrt{h}$$

find expression for h.

Marks

[33] **1.** **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box, but at most 1 mark will be given for incorrect answers. Unless otherwise stated, simplify your answer as much as possible.

(a) Evaluate $\int (2y+1)^5 \, dy$.

Answer

(b) Evaluate $\int_{-1}^{0} (2x - e^x) \, dx$.

Answer

(c) Express $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$ as a definite integral. *Do not* evaluate this integral.

Answer

Continued on page 3

(d) Write down the Simpson's Rule approximation S_4 for $\int_0^4 \frac{1}{1+x^3}\, dx$. You may leave your answer expressed as a sum of fractions.

Answer

(e) Calculate the volume of the solid obtained by rotating the region above the x-axis, below the curve $y = (\sin x)/x$, and between the lines $x = \pi/2$ and $x = \pi$ about the y-axis.

Answer

(f) Write the *form* of the partial-fraction decomposition for

$$\frac{10}{(x+1)^2(x^2+9)}$$

Do not determine the numerical values of the coefficients.

Answer

(g) An exponentially distributed continuous random variable X has probability density function $f(x) = ke^{-kx}$, for $x \geq 0$, where k is a positive constant. The *median* value of X equals 10. Find k.

Answer

Continued on page 4

(h) Find the length of the curve $y = 1 + (2/3)x^{3/2}$, $0 \le x \le 1$.

Answer

(i) Evaluate $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$.

Answer

(j) Find the first three nonzero terms in the power series representation in powers of x (i.e. the Maclaurin series) for $\displaystyle\int_0^x t\cos(t^3)\,dt$.

Answer

(k) Let $f(x) = \displaystyle\int_{e^x}^0 \cos^3 t\,dt$. Find $f'(x)$.

Answer

Continued on page 5

Full-Solution Problems. In questions 2–6, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, simplification of answers is not required.

[20] **2.** (a) [5] Sketch the bounded region that lies between the curves $y = 4 - x^2$ and $y = (x-2)^2$, and find its area. (Place only your answer for the area in the answer box.)

Answer

(b) [5] Find the numbers b such that the average value of the function $f(x) = 3x^2 - 6x + 2$ on the interval $[0, b]$ is equal to 0.

Answer

Continued on page 6

(c) [4] Set up, but do not evaluate, a definite integral for the volume of the solid obtained by rotating the region between $y = 0$ and $y = \sin x$, for $0 \le x \le \pi$, about the line $y = 1$.

Answer

(d) [6] Let R be the region under the curve $y = e^{-x}$ and above the x-axis, for $0 \le x \le 1$. Find the x-coordinate of the centroid (centre of mass) of R.

Answer

Continued on page 7

[20] **3.** Evaluate the following integrals.

(a) [4]

$$\int \frac{x}{\sqrt{1-x^4}}\,dx$$

Answer

(b) [6]

$$\int_0^1 \frac{2x+3}{(x+1)^2}\,dx$$

Answer

Continued on page 8

(c) [6]

$$\int \frac{dx}{\sqrt{x^2+2x+5}}$$

Answer

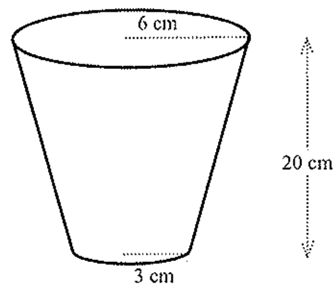
(d) [4]

$$\int (\cos^3 x)(\sin^4 x)\ dx$$

Answer

Continued on page 9

[8] **5.** A paper cup has the shape depicted below. All of its horizontal cross sections are circles; the radius of the cup's bottom is 3 cm and the radius of its top is 6 cm. The cup is full of Cona Cola, which has a density of 1000 kg/m^3. The Cona Cola is drunk through a vertical straw that extends 10 cm above the top of the cup and reaches the bottom of the cup. Express as an explicit definite integral the work performed in drinking all the cola. Do *not* evaluate this integral. For the acceleration due to gravity use the value 9.8 m/sec^2.



Answer

Continued on page 11

[7] **6.** The population of fish in a lake is m million, where $m = m(t)$ varies with time t (in years). The number of fish is currently 2 million.

(a) [3] Suppose m satisfies the logistic-growth differential equation

$$\frac{dm}{dt} = 16m\left(1 - \frac{m}{4}\right)$$

When will the number of fish equal 3 million? You may use the fact that the general solution to the logistic-growth differential equation $y' = ky(1 - (y/K))$ is $y = K/(1 + Ae^{-kt})$, where A is a constant.

Answer

(b) [4] Suppose instead that (because of fishing by humans) m satisfies

$$\frac{dm}{dt} = 16m\left(1 - \frac{m}{4}\right) - 12$$

Will the fish population ever equal 3 million? You must give justification for your answer.

Answer

The End