Math 101
April 9, 2010

Modeling with Differential Equations

I. Population growth.

~~Roughly~~

Roughly speaking, rate of growth of a bacteria culture is linearly proportional to number of bacteria present:

$P(t)$ = # of bacteria in culture at time t.

Then

$$\boxed{\frac{dP}{dt} = kP}$$

where k is some constant (depending on type of bacteria).

This boxed eq. is a simple (but important) differential equation.

Might ask to solve the differential equation. In other words, we look for function $P(t)$ satisfying $\frac{dP(t)}{dt} = kP(t)$.

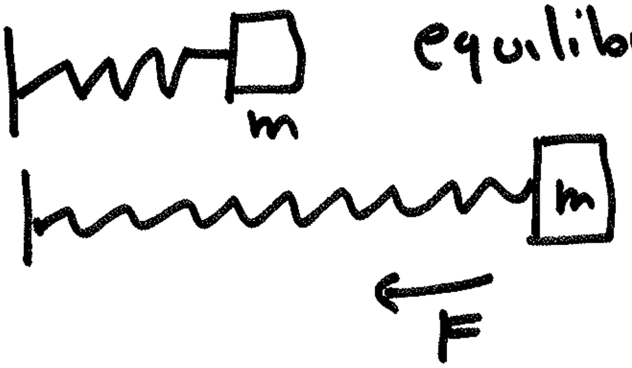
In this case its easy to guess a solution.

<u>Prop</u> 1 Set $P(t) = Ce^{kt}$ where C is a constant. Then $\frac{dP}{dt} = kP(t)$.

<u>Pf</u> $\frac{dP}{dt} = Cke^{kt} = kCe^{kt} = kP.$

<u>Remark 2</u> In fact, if P is <u>any</u> sol'n of $\frac{dP}{dt} = kP$ then $P = Ce^{kt}$ for some constant C.
This is not hard to prove but I'll skip it for now.

2. Model for spring.



equilibrium

displacement x units to right.

Recall that Hooke's law says that force F on object attached to a spring is

$$\boxed{F = -kx}$$

where x is displacement from equilibrium and k is a constant depending on spring.

Now

$$F = (\text{mass}) \times (\text{acceleration})$$
$$= m \frac{d^2x}{dt^2}$$

So Hooke's law gives differential equation

$$\boxed{m\frac{d^2x}{dt^2} = -kx}$$

— or equivalently —

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

This is a <u>second-order</u> differential equation: second because the second derivative of x appears.

The diff. eq. $\frac{dP}{dt} = kP$ is first order.

<u>Thm</u> Let $x(t)$ be a sol'n of

$$\frac{d^2x}{dt^2} = \frac{-k}{m}x, \qquad k \neq 0.$$

Then

$$x(t) = A\sin\left(t\sqrt{\frac{k}{m}}\right) + B\cos\left(t\sqrt{\frac{k}{m}}\right)$$

for some constants $A, B \in \mathbb{R}$.

<u>Sketch</u> Suppose $x(t) = \sin\left(t\sqrt{\frac{k}{m}}\right)$.

Then $x'(t) = \sqrt{\frac{k}{m}}\cos\left(t\sqrt{\frac{k}{m}}\right)$

$$x''(t) = -\sqrt{\frac{k}{m}}\sqrt{\frac{k}{m}}\sin\left(t\sqrt{\frac{k}{m}}\right)$$

$$= \frac{-k}{m}\sin\left(t\sqrt{\frac{k}{m}}\right).$$

The A's and B's don't really matter.

Uniqueness is harder.

3. General Differential Equations.

- In general, a diff. eq. is an equation containing an unknown fn and one or more of its derivatives.
- The order of a differential eq. is the order of the highest derivative of appearing.
- A function f is a solution of the diff. eq. if the diff. eq. is satisfied by substituting in $f(x)$.

Ex $\quad y' = xy \quad$ is a 1^{st} order diff. eq.

The function $y = e^{\frac{x^2}{2}}$ is a soln

because $\left(e^{\frac{x^2}{2}}\right)' = \left(\frac{2x}{2}\right) e^{\frac{x^2}{2}} = x e^{\frac{x^2}{2}}$.

- to <u>solve</u> a diff. eq. means to find all solutions.

Usually we don't want to solve a diff. eq. We just want to find a solution satisfying a particular physical property. ~~Usual~~ Often this is a condition for what the sol'n is at time 0. Called an <u>initial condition</u>.

<u>Ex</u> Find sol'n of $\frac{dP}{dt} = 5P$ satisfying $P(0) = 1$.

<u>Sol'n</u> General sol'n is $P(t) = Ce^{5t}$.

So $P(0) = C \Rightarrow$ we take $C = 1$. So answer is $\boxed{P(t) = e^{5t}}$.

4. Separable Equations.

Def A separable diff. eq. is one of the form

$$\boxed{\frac{dy}{dx} = g(x)\, f(y)}\,.$$

If $f(y) \neq 0$ then we can set $h(y) = \frac{1}{f(y)}$ and write it as

$$\boxed{\frac{dy}{dx} = \frac{g(x)}{h(y)}\,.}$$

This makes it easy to solve by multiplying out

8.5

$$\frac{dy}{dx} = \frac{g(x)}{h(x)} \implies$$

$$h(y)\,dy = g(x)\,dx$$

$$\implies \int h(y)\,dy = \int g(x)\,dx$$

If we can integrate these, we get an eq. of the form

$$H(y) = G(x) + C$$

where H is an anti-derivative of h
G " " " g.

Ex Find a sol'n of the diff. eq.

$$\boxed{y' = \frac{1}{2}(y^2-1)}$$

Satisfying $y(0)=2$ as initial condition.

Sol The eq. is seperable b/c it only involves y. To solve it we write

$$\frac{dy}{dx} = \frac{1}{\left(\frac{2}{y^2-1}\right)} \Rightarrow$$

$$\frac{2}{y^2-1}\,dy = dx \Rightarrow$$

$$\int \frac{2}{y^2-1}\,dy = \int dx.$$

Now $\frac{2}{y^2-1} = \frac{1}{y-1} - \frac{1}{y+1}$

as you can see using partial fractions.

So get

$$\int \frac{2}{y^2-1}\,dy = \ln(y-1) - \ln(y+1)$$
$$= \int dx = x + C$$

$\Rightarrow \ln\left(\frac{y-1}{y+1}\right) = x + C$ for some constant C.

Now exponentiate both sides. Get

$\frac{y-1}{y+1} = e^x e^C = \alpha e^x$ where $\alpha = e^C$ is non-zero constant.

Now solve for y.

Get

$$y - 1 = \alpha e^x y + \alpha e^x$$

$$\Rightarrow (1 - \alpha e^x) y = \alpha e^x + 1$$

$$\Rightarrow \boxed{y = \frac{1 + \alpha e^x}{1 - \alpha e^x}}.$$

Can check that this is sol'n of

$y' = \frac{1}{2}(y^2 - 1)$.

We want $y(0) = 2$. So sub

$$y(0) = \frac{1+\alpha}{1-\alpha} = 2 \Rightarrow 1 + \alpha = 2 - 2\alpha$$

$$\Rightarrow 3\alpha = 1$$

$$\Rightarrow \alpha = \frac{1}{3}.$$ So

answer is

$$\boxed{y = \frac{1 + \frac{1}{3}e^x}{1 - \frac{1}{3}e^x} = \frac{3 + e^x}{3 - e^x}}$$

Ex Find general solution of

$$y' = xy$$

Using the method of seperable diff. equations.

$$\frac{dy}{dx} = xy \Rightarrow \frac{dy}{dx} = \frac{x}{\left(\frac{1}{y}\right)}$$

$$\Rightarrow \frac{1}{y}\,dy = x\,dx \Rightarrow \int \frac{dy}{y} = \int x\,dx$$

$$\Rightarrow \ln y = \frac{x^2}{2} + C$$

$$\Rightarrow y = \exp\left(\frac{x^2}{2} + C\right)$$

$$= e^c e^{\frac{x^2}{2}}$$

$$= \alpha e^{\frac{x^2}{2}} \text{ where } \alpha \neq 0.$$

$$= y_0 e^{\frac{x^2}{2}}.$$