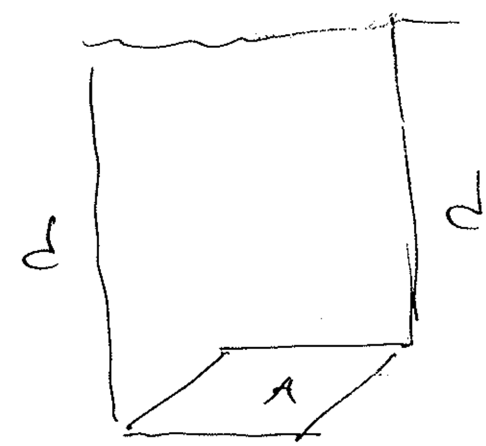
Math 101
March 7, 2010

Apps to Physics + Engineering

(1)

I Water Pressure

Suppose plate with area $A\ m^2$ is submerged in fluid of density $\rho\ \frac{kg}{m^3}$ at depth of d meters. What is the force of fluid on plate? What is the pressure?



Sol Answer is found by realizing that force is caused by weight of fluid above the plate.

$$V = (\text{Volume of fluid}) = Ad$$
$$M = (\text{mass of fluid}) = \rho V = \rho A d$$
$$F = \left(\begin{matrix}\text{force of gravity}\\ \text{on fluid}\end{matrix}\right) = Mg = \rho g A d$$

$$\boxed{g = 9.8 \frac{m}{sec^2}}$$

So

$$\boxed{F = \rho g A d}$$

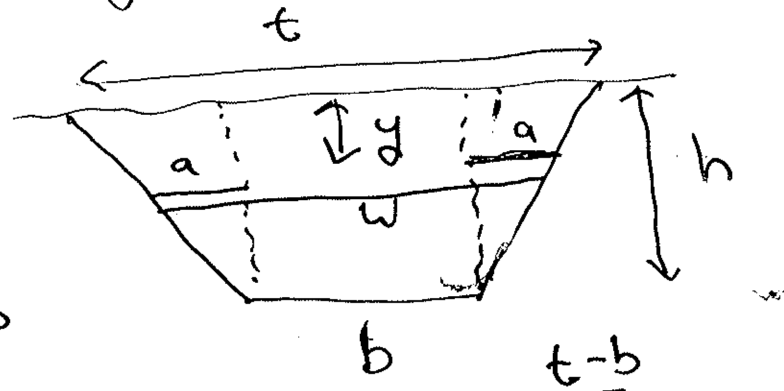
Pressure is defined as force per square meter. So

$$\boxed{P = \frac{\rho g A d}{A} = \rho g d}$$

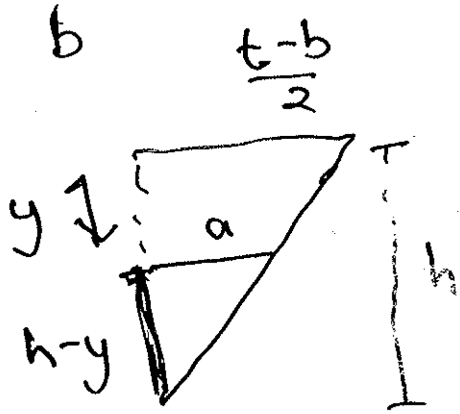
<u>Principle</u> At any point in liquid, pressure is same in all directions.

<u>Thm</u> $P = \rho g d$ where ρ = density, g = acceleration of gravity, d = depth.

Ex A dam has shape of trapezoid with height h, width t a top, and width b at bottom. Suppose water is filled in dam up to the top. Find force on dam due to hydrostatic pressure.



Sol Lets figure out width of dam at depth y. To do this use similar triangles in picture to right.



Have

$$\frac{a}{h-y} = \frac{\frac{t-b}{2}}{h}$$

$$\Rightarrow a = \frac{t-b}{2h}\cdot(h-y)$$

Dam has width $w = b + 2a$ at depth y

So we have

$$\boxed{w = b + \frac{t-b}{h}\cdot(h-y)}$$

Now at depth y we have

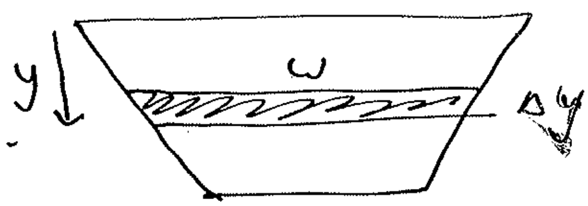
$$P = \rho g y$$

So on a cross-section of dam of width w and thickness Δy we have

$$\Delta F = P \cdot \overbrace{w \Delta y}^{\text{area}} = \rho g y w \Delta y$$

$$= \rho g y \left(b + \frac{t-b}{h}(h-y)\right) \Delta y .$$

The force on whole dam found by integrating.



So

$$\boxed{F = \int_0^h \rho g y \left(b + \frac{t-b}{h}(h-y)\right) dy}$$

I'll skip algebra now and tell answer.

It's

$$\boxed{F = \rho g h^2 \left(\frac{t + 2b}{6}\right)}$$

Moment of Inertia + Center of Mass

Observation (Archimedes)

Put two masses m_1 and m_2 at distances d_1, d_2 from a point. They balance if

$$\boxed{m_1 d_1 = m_2 d_2}.$$

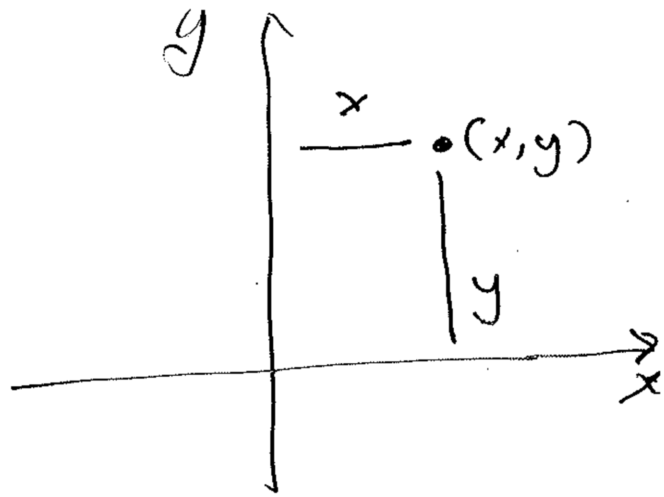
Today we understand this in terms of torque. Torque produced by gravity on m_1 is $m_1 d_1 g$ in one direction and $m_2 d_2 g$ on m_2 in other. The balance when torques are equal + opposite.

<u>Def</u> Let m be mass at point $P = (x, y) \in \mathbb{R}^2$. <u>Moment of inertia</u> of m about y-axis is

$$\boxed{M_y = m x.}$$

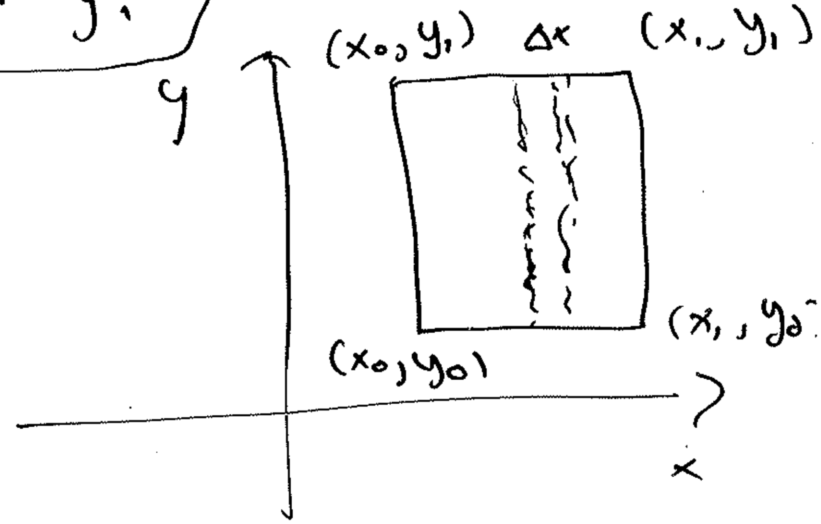
About x-axis is

$$\boxed{M_x = m y}.$$



Want to use this mainly to compute moment of inertia for regions R in plane considered to have uniform density!

<u>Problem</u> A rectangle has vertices (x_0, y_0) and (x_1, y_1) as shown. Think of rectangle R as having density 1. Compute M_x, M_y.



<u>Sol</u> To compute M_y break rectangle into smaller rectangles with height

$h = y_1 - y_0$

and thickness Δx as shown. Get

$$M_y = \int_{x_0}^{x_1} x(y_1 - y_0)\,dx$$

$$= \frac{x^2}{2}(y_1 - y_0)\Big|_{x_0}^{x_1} = \frac{(x_1^2 - x_0^2)}{2}(y_1 - y_0)$$

$$= \frac{x_1 + x_0}{2}\underbrace{(x_1 - x_0)(y_1 - y_0)}_{A} = \bar{x}A$$

Where

A = area of rectangle

$\bar{x}$ = average x coord

$= \dfrac{x_1 + x_0}{2}$

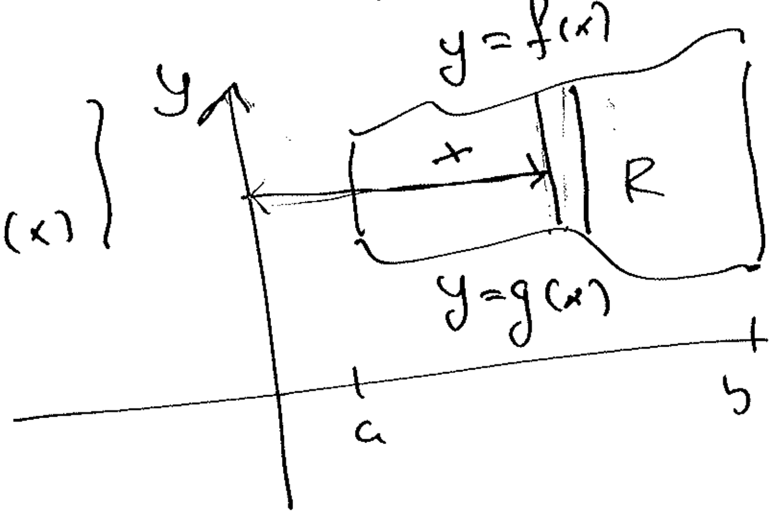
Clearly same works for M_x so

$$\boxed{\begin{aligned} M_y &= \bar{x} A \\ M_x &= \bar{y} A \end{aligned}}$$

$\bar{y} = \dfrac{y_1 + y_0}{2}$

Problem 2 Suppose R is region

$$R = \left\{ (x,y) : \begin{array}{l} x \in [a,b], \\ g(x) \le y \le f(x) \end{array} \right\}$$



Compute M_x, M_y with density 1.

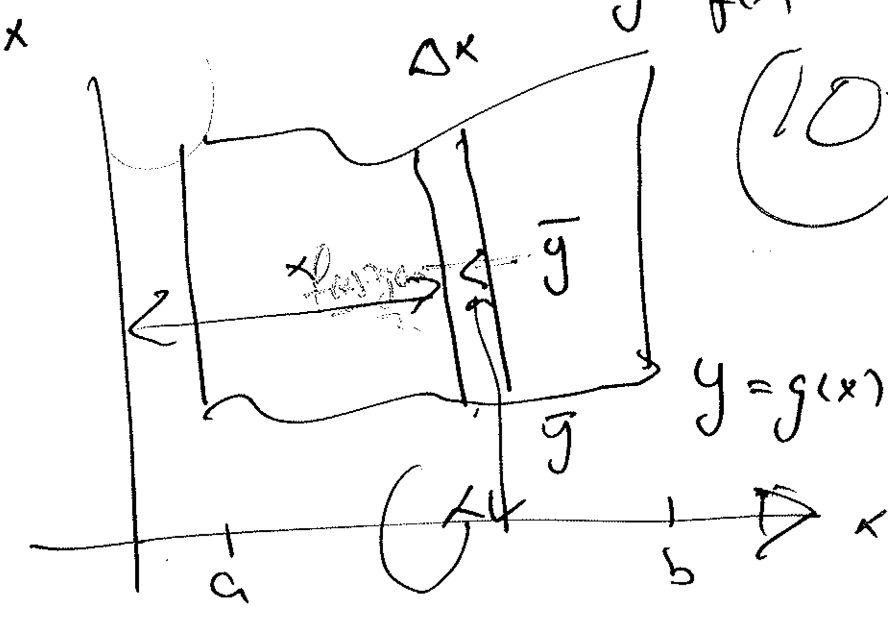
Sol $\boxed{M_y = \int_a^b x\,(f(x) - g(x))\,dx}$

That's easy to see by same trick as before.

To Compute M_x

(10)

Note that M_x for small rectangle with width Δx and and average position x is



$$\Delta M_x \simeq \bar{y}\Delta A = \frac{f(x)+g(x)}{2}\Delta A$$

$$= \frac{f(x)+g(x)}{2}\left(f(x)-g(x)\right)\Delta x = \frac{f(x)^2-g(x)^2}{2}\Delta x$$

$$\Rightarrow \boxed{M_x = \frac{1}{2}\int_a^b \left(f(x)^2 - g(x)^2\right) dx}$$

Def Center of mass of region R

is $\boxed{\bar{x} = \frac{M_y}{A}}$ $\boxed{\bar{y} = \frac{M_x}{A}}$. | A = area of R.

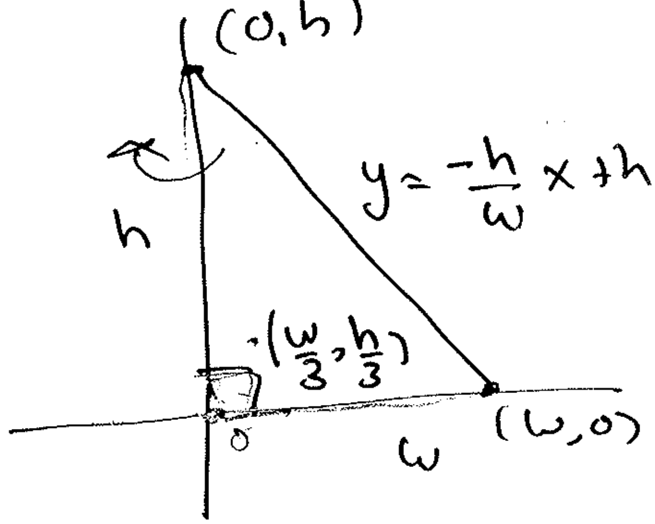
Property 1 Center of mass (or centroid) is point $P = (\bar{x}, \bar{y})$ s.t. M_x, M_y could be computed by putting all mass at point P.

Property 2 If we shift region R then center of mass shifts.

$P = (\bar{x}, \bar{y})$ is called center of mass or centroid of R.

Centroid is point where region balances.

Ex 1 Compute Center of mass of right triangle in picture.



Sol

$$M_y = \int_0^w x\left(-\frac{h}{w}x + h\right)dx$$

$$= -\frac{h}{w}\frac{x^3}{3} + \frac{hx^2}{2}\Big|_0^w$$

$$= -\frac{hw^2}{3} + \frac{hw^2}{2} = \frac{hw^2}{6}. \qquad \frac{\frac{hw^2}{6}}{\frac{hw}{2}} = \frac{w}{3}$$

$$A = \frac{hw}{2} \Rightarrow \boxed{\bar{x} = \frac{M_y}{A} = \frac{w}{3}}$$

$$\Rightarrow \boxed{\bar{y} = \frac{h}{3}}$$

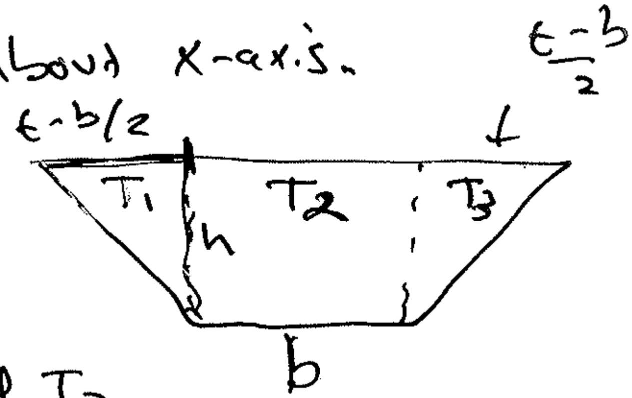
$$P = \text{centroid} = \left(\frac{w}{3}, \frac{h}{3}\right)$$

We had

$$F = \int \rho g\, y \left(b + \frac{t-b}{h}(h-y) \right) dy.$$

Don't want to do it.

$\Rightarrow$ $F = \rho g M_x$ where M_x = moment of inertia of trapezoid about x-axis.



$$M_x = M_{x,1} + M_{x,2} + M_{x,3}$$

$M_{x,1}$ = moment of inertia of T_1; $M_{x,2}$ of T_2; $M_{x,3}$ of T_3

$$= \left(\frac{t-b}{2}\right)\frac{h}{2}\cdot\frac{h}{3} + bh\cdot\frac{h}{2} + \frac{t-b}{2}\,\frac{h}{2}\cdot\frac{h}{3}$$

$$= h^2\left(\frac{t-b}{12} + \frac{b}{2} + \frac{t-b}{12} \right) = h^2\left(\frac{t-b}{6} + \frac{3b}{6}\right.$$

$$= h^2\left(\frac{t+2b}{6}\right)$$

$$\Rightarrow F = \rho g M_x = \boxed{\rho g\, h^2 \left(\frac{t+2b}{6}\right)}$$