

Math 422  
Problem Set 5  
Due Friday, November 28  
Patrick Brosnan, Instructor

1. Read chapters 3-5 of Milne's *Fields and Galois Theory*.
2. Do Ex. 3-1 to 3-3 in the Milne's book.
3. Do the starred exercises at the end of Chapter 4 of Milne.
4. Do exercise 5-1 in Milne. (Hint: Use pages 38-39.)
5. Suppose  $G$  is an abelian group generated by two elements  $x$  and  $y$ . Suppose that  $18x - 8y$  and  $60x - 26y = 0$ . Show that  $G$  is finite. Suppose that  $G$  is not cyclic. Then show that either  $G = C_2 \times C_2$  or  $C_2 \times C_6$  where  $C_n$  denotes the cyclic group with  $n$  elements.
6. Suppose that  $T$  be an endomorphism of a finite dimensional vector space  $V$  over a field  $F$ . We say that  $V$  is *cyclic* (with respect to  $T$ ) if there exists a vector  $v \in V$  such that  $V$  contains no proper  $T$ -invariant subspace  $W$  with  $v \in W$ . Show that there are polynomials  $p_1, \dots, p_r \in F[x]$  such that  $p_1 | p_2 | \dots | p_r$  and  $T$ -invariant cyclic subspaces  $(V_i)_{i=1}^r$  of dimension  $\deg p_i$  such that  $V = \bigoplus V_i$  and the restriction of  $p_i(T)$  to  $V_i$  is 0. Compute these polynomials and subspaces in the case that

$$T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$