Exchange, basis and dot product **Name:**

In class worksheet for Math 223

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Please take home and return by February 11.

Problem 1. (16 points) Let $W \subset \mathbf{R}^3$ denote the set of all vectors $(x, y, z) \in \mathbf{R}^3$ such that $x + y + z = 0$. Prove that W is a subspace of $\mathbf{R}^3$ and $\dim W \leq 2$. In fact, $\dim W = 2$. Is this intuitively obvious? Find a basis $\beta = \{\mathbf{v}, \mathbf{w}\}$ for W. It's ok to do some guessing and checking to find an answer. Now find a vector $\mathbf{u} \in \mathbf{R}^3$ such that $\gamma = \{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ is a basis for $\mathbf{R}^3$.

Solution: To see that W is a subpace, take $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W$ and $\lambda \in \mathbf{R}$. Then $x_1 + x_2 + y_1 + y_2 + z_1 + z_2 = x_1 + y_1 + z_1 + x_2 + y_2 + z_2 = 0 \Rightarrow (x_1, y_1, z_1) + (x_2, y_2, z_2) \in W$ and $\lambda x_1 + \lambda y_1 + \lambda z_2 = \lambda(x_1 + y_1 + z_1) \Rightarrow \lambda(x_1, y_1, z_1) \in W$. Thus W is closed under scalar multiplication and addition. It is obvious that $(0, 0, 0) \in W$ since $0 + 0 + 0 = 0$. Thus W is a subspace of $\mathbf{R}^3$.

Since $(1, 0, 0) \notin W$, $W \neq \mathbf{R}^3$. Therefore $\dim W < 3 = \dim \mathbf{R}^3$. So $\dim W \leq 2$. It is intuitively obvious that $\dim W = 2$ because the picture of W in $\mathbf{R}^3$ is a plane. To prove it, note that $\mathbf{v} := (1, -1, 0)$ and $\mathbf{w} := (0, 1, -1)$ are 2 linearly independent vectors in W. Therefore $\dim W \geq 2$ and it follows that $\dim W = 2$ with $\{\mathbf{v}, \mathbf{w}\}$ as a basis.

Now take $\mathbf{u} = (1, 0, 0)$. Since $\mathbf{u}$ is not in W, $\mathbf{u}$ is not in the span of $\mathbf{v}$ and $\mathbf{w}$. Therefore $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent in $\mathbf{R}^3$. Since $\dim \mathbf{R}^3 = 3$, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbf{R}^3$.

Problem 2. (16 points) Suppose $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis for a vector space V over a field F. Let λ be a non-zero element of F. Show that the following are also bases:

1. $\{\mathbf{a}, \mathbf{a}+\mathbf{b}, \mathbf{c}\}$,
2. $\{\lambda\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

Solution: 1. Suppose $\mathbf{v} \in V$, then we can write $\mathbf{v} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ with $x, y, z \in F$. Therfore we also have $\mathbf{v} = y(\mathbf{a}+\mathbf{b}) + (x-y)\mathbf{a} + z\mathbf{c}$. It follows that $\{\mathbf{a}, \mathbf{a}+\mathbf{b}, \mathbf{c}\}$ spans V. Since V has dimension 3, it folllows that $\{\mathbf{a}, \mathbf{a}+\mathbf{b}, \mathbf{c}\}$ is a basis for V.

2. Again suppose $\mathbf{v} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$. Then we have $\mathbf{v} = (x/\lambda)(\lambda\mathbf{a}) + y\mathbf{b} + z\mathbf{c}$. So $\{\lambda\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ spans V.

Problem 3. (20 points) The *dot product* of two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ in $\mathbf{R}^n$ is defined to be $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v_i w_i$. Compute $(1, 2) \cdot (2, 3)$.

Now suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^n$, and $\alpha \in \mathbf{R}$. Prove the following properties of the dot product

1. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and equality occurs if and only if $\mathbf{u} = 0$;
2. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;
3. $(\alpha\mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$.

The *norm* of a vector $\mathbf{u} \in \mathbf{R}^n$ is defined to be $|\mathbf{u}| := \sqrt{\mathbf{u} \cdot \mathbf{u}}$. Compute $|(1, 2)|$. Explain why, for $\mathbf{u}$ in $\mathbf{R}^2$, $|\mathbf{u}|$ is the distance from $(0, 0)$ to $\mathbf{u}$.

Solution: 1. Write $\mathbf{u} = (u_1, \ldots, u_n)$. Then $\mathbf{u} \cdot \mathbf{u} = \sum u_i^2$. Since the square of a real number is non-negative and the the non-negative real numbers are closed under addition, it follows that $\mathbf{u} \cdot \mathbf{u} \geq 0$.

2. With $\mathbf{u}$ as above and $\mathbf{v} = (v_1, \cdots, v_n)$, we have $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i = \sum_{i=1}^n v_i u_i = \mathbf{v} \cdot \mathbf{u}$.

3. This is obvious so I skip the proof.

The norm is the distance to the origin by the Pythagorean theorem.

Problem 4. (16 points) Suppose $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$. Show that

$$\mathbf{u} \cdot \mathbf{v} = \frac{|(\mathbf{u}+\mathbf{v})|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2}{2}.$$

To do this, the best strategy is to write $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ and then expand out. This is interesting because it allows us to express the dot product (something you might not have seen before) completely in terms of length which you probably do know about from geometry. I'll use it and the law of cosines from high-school geometry to prove the following result.

Theorem. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ *where* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

Solution: *We have*

$$\begin{aligned}\frac{|(\mathbf{u}+\mathbf{v})|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2}{2} &= \frac{(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) - \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v}}{2} \\ &= \frac{\mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v}}{2} \\ &= \mathbf{u} \cdot \mathbf{v}\end{aligned}$$

Problem 5. (16 points) Suppose $V \subset \mathbf{R}^3$ has dimension 2. Show that there exists a vector $\mathbf{w} \in \mathbf{R}^3$ such that

$$V = \{\mathbf{v} \in \mathbf{R}^3 \mid \mathbf{v} \cdot \mathbf{w} = 0\}.$$

Show that $\mathbf{w}$ is unique up to multiplication by a non-zero scalar. Hint: Pick a basis

$$\{(x_1, y_1, z_1), (x_2, y_2, z_2)\}$$

for V. Then solve explicitly for w in terms of the basis.

Problem 6. (16 points) Let $x, y \in \mathbf{R}$. Show that the vectors $(1,1,1), (1,x,y)$ and $(1,x^2,y^2)$ form a basis of $\mathbf{R}^3$ if and only if $1, x$ and y are all distinct.

Problem 7. (Bonus 10 points) Let $W \subset \mathbf{R}^n$. The *orthogonal complement* of W is the set

$$W^\perp := \{x \in \mathbf{R}^n \mid \forall y \in W \; x \cdot y = 0\}.$$

Show that $W^\perp$ is a subspace of $\mathbf{R}^n$ and that

$$\mathbf{R}^n = W \oplus W^\perp.$$

Let $W = \langle(1,2,3)\rangle \subset \mathbf{R}^3$. Find a basis for $W^\perp$. Draw a picture of W and $W^\perp$ in $\mathbf{R}^3$.

Solution: Suppose $a, b \in W^\perp$ and $\lambda \in \mathbf{R}$. Then, for all $w \in W$, $(a+b)\cdot w = a \cdot w + b \cdot w = 0 + 0 = 0$, and $(\lambda a) \cdot w = \lambda(a \cdot w) = \lambda 0 = 0$. This shows that $W^\perp$ is closed under addition and scalar multiplication. Since $0 \cdot w = 0$ for all $w \in W$, we see that $W^\perp$ is non-empty. Thus $W^\perp$ is a subspace.

Suppose $v \in W \cap W^\perp$. Then $v \cdot v = 0$. Therefore $v = 0$ by Problem 3. Thus $W \cap W^\perp = \{0\}$.

I will work the example before proving that $\mathbf{R}^n = W \oplus W^\perp$. We have $W = \langle(1,2,3)\rangle$. I claim that a basis for $W^\perp$ is then $\{(3,0,-1),(2,-1,0)\}$. It is easy to see that these are a linearly independent and in $W^\perp$. Since $W^\perp$ is clearly not all of $\mathbf{R}^n$, we must have $\dim W^\perp \le 2$. Thus, $W^\perp = \langle(3,0,-1),(2,-1,0)\rangle$.

Now, I prove that $W + W^\perp = \mathbf{R}^n$. Pick a basis $w_1, \ldots, w_r$ for W and define a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^r$ by $T(v) = (v \cdot w_1, \ldots, v \cdot w_r)$. It is clear that $W^\perp \subset N(T)$ and, if $v \in N(T)$ then $v \cdot w_i = 0$ for all i; therefore for $w = \sum a_i w_i \in W$, $v \cdot w = 0$. Thus $v \in W^\perp$. Hence, $N(T) = W^\perp$. It follows from the rank-nullity theorem that $\dim \mathbf{R}^n = \dim W^\perp + \dim T(\mathbf{R}^n)$. Since $\dim T(\mathbf{R}^n) \le r$, it follows that $\dim W^\perp \ge n - r$. On the other hand, from the Hausdorff dimension formula and the fact that $W \cap W^\perp = \{0\}$, we know that $\dim W^\perp \le n - r$. Therfore $\dim W^\perp = n - r$ and, using the Hausdorff dimension formula again, we see that $\dim W + W^\perp = n$. It follows that $W + W^\perp = \mathbf{R}^n$.

Problem 8. (Ungraded) Make sure you have done all practice problems on HW 1 and HW 2. For more practice, do any mandatory problems on HW 2 that you have not done yet.

Problem 9. (Ungraded Essay.) Do you think this problem set helped you to understand basis and spanning in a geometric way? When you started the problem set, had you already seen the dot product? My reason for introducing the dot product here was to connect the notions we have been learning in linear algebra with the more geometrical notion of the dot product. Let me know if you found this helpful or would prefer another approach.