Replacement Theorem
Math 223 Notes by Patrick Brosnan

The replacement theorem was proved in class on Monday, February 2. As it is quite important, I wanted to write notes on it to make clear the facts that we are using.

Notation. For a set S, I write $|S|$ for the number of elements of S and $|S| = \infty$ if S is infinte. For example, $|\{1, 2, 3\}| = 3$ and $|\emptyset| = 0$. If V is a vector space over a field F then a relation $\sum_{i=1}^{n} a_i v_i = 0$ with $a_i \in F$, $v_i \in V$ is called a *dependence relation* if not all a_i are 0.

Proposition 1. *Let V be a vector space over a field F. Let $S \subset V$ and let W be a subspace of V. Then $\langle S \rangle \subset W$ iff $S \subset W$.*

Proof. Exercise. (This was proved in class in the notes.)

Theorem 2. *Let V be a vector space over a field F. Let $I \subset V$ be independent and let $v \in V$ with $v \notin I$. Then $I \cup \{v\}$ is dependent $\Leftrightarrow v \in \langle I \rangle$.*

Proof. $\Rightarrow$: Suppose $I \cup \{v\}$ is dependent. Then there are $w_1, \ldots, w_n \in I$ and scalars $a_0, a_1, \ldots, a_n \in F$ not all 0 such that

$$a_0 v + \sum_{i=1}^{n} a_i w_i = 0.$$

We cannot have $a_0 = 0$ because this would give a non-trivial dependence relation among the w_i contradicting the assumption that I is independent. Therefore $a_0 \neq 0$. Thus

$$v = \sum_{i=1}^{n} (\frac{-a_i}{a_0}) w_i.$$

Hence $v \in \langle I \rangle$.

$\Leftarrow$: Suppose $v \in \langle I \rangle$. Then $v = \sum_{i=1}^{n} a_i w_i$ for some $a_i \in F$. Therefore

$$-v + \sum_{i=1}^{n} a_i w_i = 0.$$

This is a non-trivial dependence relation showing that $\{v\} \cup I$ is dependent.

Definition. Suppose X is a collection of sets. A set $S \in X$ is said to be *maximal in X* if no element $T \in X$ properly contains S. For example, suppose $X = \{\emptyset, \{1\}, \{1, 2\}, \{3\}, \{4, 5\}\}$. Then $\{1, 2\}, \{3\}$ and $\{4, 5\}$ are all maximal by $\{1\}$ and $\emptyset$ are not. If X is finite, then it is fairly obvious that X contains maximal elements.

Theorem 3. *Let V be a vector space over a field F. Suppose $G \subset V$ is a set spanning V. Let $\mathcal{I}$ denote the collection of all independent subsets of G. Then any maximal element of $\mathcal{I}$ is a basis for V.*

Proof. Suppose B is a maximal independent subset in G. To show that B is a basis, we only need to show that B spans V. If not, then $\langle G \rangle = V$ is not contained in $\langle B \rangle$. From Proposition 1, it follows that there is a $w \in G$ such that $w \notin \langle B \rangle$. But then $\{w\} \cup B$ is independent. This contradicts the maximality of B.

Corollary 4. *Let V be a vector space over a field F. Suppose $G \subset V$ is a finite set spanning V. Then there is a subset $B \subset G$ which is a basis for V.*

Proof. Since G is finite, we can find a maximal element of the (finite) set $\mathcal{I}$ of independent subsets of G.

The following Lemma corrects a mistatement I made in class stating the replacement theorem. Recall that two sets S and T are *disjoint* if $S \cap T = \emptyset$. In class I forgot to say that H and I are assumed disjoint.

Lemma 5. *Suppose V is a vector space over a field F. Let $I = \{v_1, \ldots, v_m\}$ be an independent subset of V and let G be a subset of V with n elements such that $\langle G \rangle = V$. Let H be a subset of G disjoint from I such that $I \cup H$ is independent. Then $|H| \leq n - m$.*

Proof. I prove this by induction on m. It is obvious for $m = 0$. Suppose then that it is true for $n = 1$. Take a $H = \{w_1, \dots, w_s\}$ to be a subset of G disjoint from I such that $I \cup H$ is independent. Set $I' = \{v_1, \dots, v_{m-1}\}$. Then, by Theorem 2, $v_m \notin \langle I' \cup H \rangle$. Thus $\langle I' \cup H \rangle \neq V$. It follows that there exists a $w_{s+1} \in G$ such that $w_{s+1} \notin \langle I' \cup H \rangle$. Thus

$$\{v_1, \dots, v_{m-1}, w_1, \dots, w_{s+1}\}$$

is independent. By induction, it follows that $s + 1 \leq n - (m - 1) = n - m + 1 \Rightarrow s \leq n - m$.

Lemma 6. *Let V, I and G be as in Lemma 5. Let $\mathcal{C}$ denote the collection of all subsets H of G such that $H \cap I = \emptyset$ and $I \cup H$ is independent. If H is maximal in $\mathcal{C}$ then $I \cup H$ is a basis.*

Proof. If not, there is a $w \in G$ such that $w \notin \langle I \cup H \rangle$. Then $\{w\} \cup I \cup H$ is independent. This contradicts the maximality of H.

Corollary 7 (Replacement). *Let V be a vector space over a field F generated by a set G with n vectors. Let I be a linearly independent subset of V with m vectors. Then there exists a subset H of G with at most $n - m$ vectors such that $I \cup H$ is a basis.*

Proof. Take a maximal H in the collection $\mathcal{C}$ above.

Example 8. Here is an example to indicate what can go wrong in Lemma 5 if I and H are not disjoint. Take $V = \mathbf{R}^2$ and let $I = \{(1, 0)\}$. So $m = 1$. Take $G = \{(1, 0), (0, 1)\}$ so $n = 2$. Then $I \cup G = G$ is independent. So we cannot take $H = G$ in Lemma 5 because it $2 > 2 - 1 = 1$. However, if we take $H = \{(0, 1)\}$ then $H \cap I$ is disjoint and we see that $I \cup H$ is a basis.