The IVT and the EVT
Math 120 Notes
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The pupose of these notes is to prove the two most difficult theorems of differential calculus, the Intermediate Value Theorem and the Extreme Value Theorem. As discussed in class, the main tool is the least upper bound property of the real numbers. We begin with a lemma which seems almost obvious (but which is really the heart of the matter).

Lemma 1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Suppose that, for all $x \in (-\infty, 0]$, $f(x) = 1$ and that, for all $x \in \mathbf{R}$, $f(x)$ is either -1 or 1. Then, $f(x) = 1$ for all $x \in \mathbf{R}$.*

Proof. To get a contradiction, suppose f is a counterexample to the Lemma. Let $S = \{x \in \mathbf{R} \mid f(x) = -1\}$. Then S is non-empty, since f is a counterexample, and S is bounded below by 0. Therefore $\inf S$ exists, and we set $c = \inf S$.

Now, since f is continuous, we can find $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < 1$. Since c is a lower bound for S, we must have $f(c - \delta/2) = 1$. However, since c is the greatest lower bound for S, there must exist $y \in [c, c + \delta)$ such that $f(y) = -1$. But then, by the triangle inequality,

$$2 = |f(y) - f(c - \delta/2)| \leq |f(y) - f(c)| + |f(c) - f(c - \delta/2)| < 1 + 1 = 2.$$

This is a contradiction. So we conclude that the lemma holds for all f. Q.E.D.

Now we apply the lemma to get another lemma which was stated in class.

Lemma 2. *Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Suppose that, for all $x \in [a, b]$, $f(x)$ is either -1 or 1. Then $f(x) = f(a)$ for all $x \in [a, b]$.*

Proof. Suppose $f : [a, b] \to \mathbf{R}$ satisfies the hypotheses of the lemma. Define a function $h : \mathbf{R} \to \mathbf{R}$ by setting

$$h(t) = \begin{cases} 1, & t \in (-\infty, 0); \\ f(a)f(t + a), & t \in [0, b - a]; \\ f(a)f(b), & t \in (b - a, \infty). \end{cases}$$

I claim that h satisfies the hypotheses of Lemma 1. To see this, first note that, since f takes only the values ± 1, so does h. Also $h(0) = f(a)^2 = 1$; thus, $h(t) = 1$ for all $t \leq 0$. It is clear that h is continuous except possiby at 0 and $b - a$. We know that h is continuous at 0 because 1 and $f(a)f(t + a)$ are both continuous and take the value 1 at $t = 0$. Similarly, h is continous at $b - a$ because $f(a)f(t + a)$ and $f(a)f(b)$ are continuous and take the value $f(a)f(b)$ at $t = b - a$.

Since h satisfies the hypotheses of Lemma 1, we must have $h(x) = 1$ for all x. Since, for $x \in [a, b]$, we have

$$f(x) = f(a)h(x - a)$$

it follows that f is constant on $[a, b]$. Q.E.D.

Theorem (IVT). *Let $f : [a, b] \to \mathbf{R}$ be a continuous function and let V be a real number between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = V$.*

Proof. Suppose, to get a contradiction, that there exists V between $f(a)$ and $f(b)$ such that, $V \notin f([a, b])$. Then define a function $h : [a, b] \to \mathbf{R}$ by

$$h(x) = \frac{f(x) - V}{|f(x) - V|}.$$

This is a well-defined function because the denominator is never 0. Moreover, $h(x) \in \{\pm 1\}$ for all $x \in [a, b]$, and $h : [a, b] \to \mathbf{R}$ is continuous since the quotient of two continous functions is continuous as long as the denominator is non-zero.

However, if $f(a) < V < f(b)$, we have $h(a) = -1$ and $h(b) = 1$ while, if $f(a) > V > f(b)$, we have $h(a) = 1$ and $h(b) = -1$. It follows that, the existence of h contradicts Lemma 2. Q.E.D.

Now, we are going to prove the extreme value theorem using a very similar stategy. Suppose f is a real-valued function whose domain contains a subset $S \subset \mathbf{R}$. We say that f is bounded on S if there exists an $M \in \mathbf{R}$ such that, for all $x \in S$, $|f(x)| \leq M$.

Lemma 3. *If S and T are two subsets of $\mathbf{R}$ and f is bounded on both S and T, then f is bounded on $S \cup T$. If S and T are two subsets of $\mathbf{R}$ such that $T \subset S$ and f is bounded on S then f is also bounded on T.*

Proof. Exercise.

Lemma 4. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $f(x) = 0$ for $x \leq 0$. Then, for all $a \in \mathbf{R}$, f is bounded on $(-\infty, a]$.*

Proof. Define a function $h : \mathbf{R} \to \mathbf{R}$ by

$$h(t) = \begin{cases} 1, & \text{if } f(x) \text{ is bounded on } (-\infty, x]; \\ -1, & \text{otherwise.} \end{cases}$$

I claim that h satisfies the hypotheses of Lemma 1. Clearly, $h(x) = 1$ for all $x \leq 0$, since f is bounded by 0 on $(-\infty, 0]$. It is also clear from the definition of h that, for all $x \in \mathbf{R}$, $h(x) = \pm 1$. Now, I claim that h is continous on $\mathbf{R}$. To prove this claim, suppose $c \in \mathbf{R}$ and pick $\epsilon > 0$. Since f is continuous at c, we can find $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < 1$. Therefore, f is bounded on $(c - \delta, c + \delta)$ by $|f(c)| + 1$. If $h(c) = 1$, then it follows from Lemma 3 that f is bounded on $(-\infty, c + \delta)$. Thus $h(x) = 1$ for all $x \in (-\infty, c + \delta)$. In particular, $|x - c| < \delta \Rightarrow |h(x) - h(c)| = 0 < \epsilon$. On the other hand, if $h(c) = -1$, then f is not bounded on $(-\infty, c]$. Since f is bounded on $(c - \delta, c + \delta)$, Lemma 3 implies that f is not bounded on $(-\infty, c - \delta]$. Thus f is not bounded on $(-\infty, x]$ for any $x \geq c - \delta$. Therefore, $h(x) = -1$ for all $x \geq c - \delta$. In particular, $|x - c| < \delta \Rightarrow |h(x) - h(c)| = 0 < \epsilon$.

Now, since h satisfies the hypotheses of Lemma 1, we must have $h(a) = 1$ for all $a \in \mathbf{R}$. Thus, by definition of h, f is bounded on $(-\infty, a]$ for all $a \in \mathbf{R}$. Q.E.D.

Proposition. *Let $f : [a, b] \to \mathbf{R}$ be a continous function. Then f is bounded on $[a, b]$.*

Proof. Define $F : \mathbf{R} \to \mathbf{R}$ by

$$F(t) = \begin{cases} 0, & t < 0; \\ f(t + a) - f(a), & t \in [0, b - a]; \\ f(b) - f(a), & t \in (b - a, \infty). \end{cases}$$

Then $F(t) = 0$ for $t \leq 0$ and F is easly seen to be continuous. Therefore F is bounded. But if F is bouunded by M, then f is bounded by $M + |f(a)|$. Thus, f is also bounded. Q.E.D.

Theorem (Extreme Value Theorem). *Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Then the set $f([a, b])$ has a maximum.*

Proof. Suppose, to get a contradiction that $f : [a, b] \to \mathbf{R}$ is a continuous function with no maximum. Set $M = \sup f([a, b])$, and define a function $h : [a, b] \to \mathbf{R}$ by

$$h(x) = \frac{1}{M - f(x)}.$$

Clearly, h is continous since, by assumption, there is no $x \in [a, b]$ such that $f(x) = M$. I claim that h is unbounded on $[a, b]$.

To verify this claim, suppose that there is an $N \in \mathbf{R}$ such that $|h(x)| \leq N$ for all $x \in [a, b]$. Without loss of generality, we can assume that $N > 0$. But then simple algebraic manipulation shows that $f(x) \leq M - 1/N < M$ for all $x \in [a, b]$. This contradicts our assumption that M was the least upper bound for f. Q.E.D.

Corollary (EVT for Minima). *Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Then the set $f([a, b])$ has a minimum.*

Proof. Excercise.