Taylor series with remainder
Notes for Math 120
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The purpose of these notes is to prove Taylor's theorem with remainder. Let I be an open interval in $\mathbf{R}$ containing a point a, and let $f : I \to \mathbf{R}$ be a function. We say that f is *n times differentiable at a* if $f^{(n)}(a)$ exists. We say that f is *n times differentiable on I* if, f is n-times differentiable at every point $a \in I$.

Now let $f : I \to \mathbf{R}$ a function that is n times differentiable at a point $a \in I$. The *n-th Taylor polynomial of f at a* is the polynomial

$$P_{n,a,f}(x) = \sum_{i=0}^{n} f^{(i)}(a) \frac{(x-a)^i}{i!}.$$

When n, a and f are understood, we simply write $P(x)$ for this polynomial.

Theorem 1. *Let $P = P_{n,a,f}(x)$ be the n-th Taylor polynomial of f at a. Then, for $i \in \{0, \dots, n\}$, $P^{(i)}(a) = f^{(i)}(a)$, while for $i > n$, $P^{(i)}(a) = 0$.*

Proof. We have

$$P^{(i)}(a) = \sum_{j=0}^{n} f^{(j)}(a) \frac{d^i}{dx^i} \frac{(x-a)^j}{j!}.$$

Now

$$\frac{d^i}{dx^i} \frac{(x-a)^j}{j!}_{|x=a} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have $P^{(i)}(a) = f^{(i)}(a)$.

Now, we want to prove Taylor's theorem with remainder. But, perhaps the idea of the proof is best seen without the extra notation of the point a and even the derivatives of the functions. So we will first prove a lemma which is, in fact, the theorem in a very special case.

Lemma 2. *Let I be an open in $\mathbf{R}$ and suppose $a, b \in I$ with $a \neq b$. Let n be a non-negative integer and suppose f is $n+1$ times differentiable on I. Suppose that $f^{(i)}(a) = 0$ for all $i \in \{0, \dots, n\}$ and that $f(b) = 0$. Then there is a c between a and b such that $f^{(n+1)}(c) = 0$.*

Proof. For $n = 0$, the statement is simply Rolle's theorem. So, to get a contradiction, assume that n is the smallest integer for which the statement does not hold. Since $f(a) = f(b) = 0$, we can find c_1 between a and b such that $f'(c_1) = 0$. But then set $g = f'$. We have $g^{(i)}(a) = 0$ for $i \in \{0, \dots, n-1\}$ and $g(c_1) = 0$. Therefore, there exists c between a and c_1 such that $f^{(n+1)}(c) = g^{(n)}(c) = 0$. Since c is between a and c_1 it is between a and b. Thus we have a contradiction.

Theorem 3. *Let $f : I \to \mathbf{R}$ be a function which is $n+1$ times differentiable for n a non-negative integer. Let $a, b \in I$, and set $P(x) = P_{n,a,f}(x)$. Then there is a c between a and b such that*

$$f(b) = P(b) + f^{(n+1)}(c) \frac{(b-a)^{n+1}}{n+1!}.$$

Proof. If $a = b$, this is obvious. So suppose $a \neq b$ and set

$$h(x) = f(x) - P(x) - \frac{(x-a)^{n+1}}{(b-a)^{n+1}} (f(b) - P(b))$$

Then $h^{(i)}(a) = 0$ for all $i \in \{0, \dots, n\}$ and $h(b) = 0$. Moreover h is $n+1$ times differentiable on I. It follows that there exists c between a and b such that $h^{(n+1)}(c) = 0$. But

$$h^{(n+1)}(c) = f^{(n+1)}(c) - (n+1)! \frac{(f(b) - P(b))}{(b-a)^{n+1}} = 0.$$

implies the conclusion of the theorem by simple algebraic manipulation.

Example. *Estimate the value of e using the Taylor's theorem with remainder for the function e^x. Take $a = 0$.*

Solution. First note that, when $a = 0$, the Taylor polymomial is, for historical reasons, traditionally called the *MacLaurin Polynomial.* Now the n-th MacLaurin polynomial for e^x is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

The theorem says that there is a c between 0 and 1 such that $e = P(1) + e^c/(n+1)!$. Since e^x is an increasing function, we have $1 \leq e^c \leq e$. Therefore

$$P_{n+1}(1) \leq e \leq P_n(1) + \frac{e}{(n+1)!}.$$

The right-hand inequality can be rearranged to give us

$$e \leq P_n(1)\frac{(n+1)!}{(n+1)! - 1}.$$

Thus we have

$$1 + 1 + \frac{1}{2!} + \cdots \frac{1}{(n+1)!} \leq e \leq [1 + 1 + \frac{1}{2!} + \frac{1}{n!}]\frac{(n+1)!}{(n+1)! - 1}.$$

Here is a table of some of the left-hand side and right-hand side values of this inequality for e: l indicates the lower bound and r the upper bound.

n	1	2	3	4	5	6
l	2.5	2.6667	2.7083	2.7167	2.7181	2.7182
r	4	3	2.7826	2.7311	2.7204	2.7186

The actual value is closer to

$$e = 2.71828182845904523536028747l.$$