Admissible normal functions and Hodge theory

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Outline

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- C smooth, projective algebraic curve,
- $D \subset C$ degree 0 divisor,
- $\mathrm{Jac}(C) =$ Jacobian of C, isomorphic to the complex torus

$$H^1(C, \mathbb{C})/(H^{1,0}(C) + H^1(C, \mathbb{Z})).$$

- the Abel-Jacobi map gives a class $\mathrm{AJ}(D) \in \mathrm{Jac}(C)$. Arises from integrating 1-forms ω over a 1-chain γ with $\partial(\gamma) = D$.

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- X smooth projective surface with (surjective) morphism $f : X \to S$, S a smooth, projective, curve.
- E.g, above situation arises when one blows up the base locus in a Lefschetz pencil.
- $S^{\mathrm{sm}} := \{s \in S : X_s \text{ smooth}\}$.
- $D \subset X$ divisor, relative degree 0 divisor over S.
- Get a family $\pi : \mathrm{Jac} \to S^{\mathrm{sm}}$ with fiber $\mathrm{Jac}(X_s)$. The divisor D gives a section $\nu : S^{\mathrm{sm}} \to \mathrm{Jac}$ of π given by $s \mapsto \mathrm{AJ}(D_s)$ of π. This is a *normal function*

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- In general, normal functions do not take values in algebraic varieties but rather in complex tori. These tori, which are called *Griffiths intermediate Jacobians*, can be understood as Ext groups in the category of mixed Hodge structures.

Theorem (Carlson)

Let $\mathbf{H}$ *be a pure Hodge structure of negative weight. Then the group* $\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}, \mathbf{H})$ *of extensions of the Hodge structure* $\mathbb{Z}(0)$ *by* $\mathbf{H}$ *is canonically isomorphic to the complex torus*

$$J(\mathbf{H}) := \mathbf{H}_{\mathbb{C}}/(F^0\mathbf{H} + \mathbf{H}_{\mathbb{Z}}).$$

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Variations of Pure Hodge structure

A variation of pure Hodge structure of weight w on a complex manifold S consists of

- a local system $\mathbf{H}$ on S,
- a filtration of the flat vector bundle $\mathbf{H}_{\mathcal{O}_S} = \mathbf{H} \otimes_{\mathbb{Z}} \mathcal{O}_S$ by holomorphic subbundles F^p.

This data is required to satisfy

- the F^p induce a pure Hodge structure of weight w on each of the fibers $\mathbf{H}_s$, $s \in S$,
- the F^p satisfy *Griffiths transversality*:

$$\nabla F^p \subset F^{p-1} \otimes_{\mathcal{O}_S} \Omega^1_S.$$

There is also a condition of *polarizability*.

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- for each k, $\mathrm{Gr}^W_k\,\mathbf{H}$ is a variation of pure Hodge structure of weight k (with rational coefficients).

The category $\mathrm{HS}(S)$ of variations of pure Hodge structure is a semi-simple abelian category. The category $\mathrm{MHS}(S)$ of variations of mixed Hodge structure over S is an abelian category.

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Admissible variation of mixed Hodge structure

But the category of variation of mixed Hodge structure is too big. It contains objects which do not come from geometry and have wild behavior at infinity. The *correct* category to work in is the category of *admissible variations of mixed Hodge structure* which we have thanks to work of Deligne, Steenbrink-Zucker and Kashiwara.

The condition of admissibility is best spelled out first on the puncture disk. So set

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- Choose a basepoint $p \in \Delta$. Then the data of local system underlying a variation of mixed Hodge structure $\mathbf{H}$ on Δ^* can be thought of as a finitely generated abelian group $H = \mathbf{H}_p$ equipped with an operator $T \in \operatorname{Aut} H$.
- By a theorem of Borel, T is quasi-unipotent. That is, there exists positive integers n, m such that $(T^n - 1)^m = 0$.
- Assume T is unipotent and set $N = \log T$. Since the weight filtration W is a local system, T and thus N preserves W.
- In the case of admissible variations one demands that N induces a further filtration M on H called the *relative weight filtration*.

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Proposition (Deligne)

Let N be a nilpotent operator on a vector space H preserving an exhaustive increasing filtration $(W_n)_{n\in\mathbb{Z}}$. Then there exists at most one filtration $M = M(N,W)$ of H such that

- $N(M_n) \subset M_{n-2}$,
- *for all k,* $N^k : \mathrm{Gr}^M_{w+k}\mathrm{Gr}^W_w H \to \mathrm{Gr}^M_{w-k}\mathrm{Gr}^W_w$ *is an isomorphism.*

- The filtration $M = M(N,W)$ is called the relative weight filtration if it exists.
- A variation $\mathbf{H}$ on Δ^* is said to be admissible relative to Δ if the relative weight filtration exists and the Hodge filtration extends to a holomorphic subbundle of the canonical extension of $\mathbf{H}\otimes_{\mathbb{Z}}\mathcal{O}_{\Delta^*}$ to Δ.

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Admissibility in General

Definition (Kashiwara)

Let S be a complex manifold contained as a Zariski open subset in a complex manifold $\bar{S}$, then a variation $\mathbf{H}$ on S is admissible relative to $\bar{S}$ if, for any disk $\Delta \subset \bar{S}$ with $\Delta^* = \Delta \cap S$, the restriction of $\mathbf{H}$ to Δ^* is admissible relative to Δ.

- Write $\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$ for the (abelian) category of admissible variations of S relative to $\bar{S}$.
- If S is quasi-projective, then the category $\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$ is the same for every smooth projective $\bar{S}$ containing S. So we just write $\mathrm{VMHS}(S)^{\mathrm{ad}}$.
- $\mathrm{VMHS}(S)^{\mathrm{ad}}$ is the full subcategory of the category of mixed Hodge modules on S consisting of smooth mixed Hodge modules.

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- Let **H** be a variation of pure Hodge structure of negative weight over a complex manifold S. By Carlson's formula, an element of $\nu \in \mathrm{Ext}_{\mathrm{VMHS}(S)}(\mathbb{Z}, H)$ gives a section of the family $J(H) \to S$. In fact, it is not hard to see that the element ν is determined by this section.

Definition (Morihiko Saito)

Write $\mathrm{NF}(S, \mathbf{H}) := \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbb{Z}, \mathbf{H})$. This is the group of *normal functions*. The subgroup $\mathrm{NF}(S, \mathbf{H})^{\mathrm{ad}} := \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}}(\mathbb{Z}, \mathbf{H})$ is the group of *admissible normal functions*.

- Every admissible normal function determines (and is determined by) a section of $J(\mathbf{H}) \to S$. For this reason it makes sense to view admissible normal functions as functions.

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Write $\mathrm{NF}(S, \mathbf{H}) := \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbb{Z}, \mathbf{H})$. This is the group of *normal functions*. The subgroup $\mathrm{NF}(S, \mathbf{H})^{\mathrm{ad}} := \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}}(\mathbb{Z}, \mathbf{H})$ is the group of *admissible normal functions*.

- Every admissible normal function determines (and is determined by) a section of $J(\mathbf{H}) \to S$. For this reason it makes sense to view admissible normal functions as functions.

Theorem (B-Pearlstein, Schnell)

Let $\nu \in \mathrm{NF}(S, \mathbf{H})^{\mathrm{ad}}$ *for* S *algebraic. Set* $Z(\nu) = \{s \in S : \nu(s) = 0\}$. *Then* $Z(\nu)$ *is a closed subvariety of* S.

- Our proof involves an analysis of the asymptotics of splittings of mixed Hodge structures which I will describe below. The main tool is a theorem of Kato, Nakayama and Usui.
- Christian Schnell has independently proved the theorem at least in the case that $\mathbf{H}$ is pure of weight -1.
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Application

- The theorem is best seen as an extension of the theorem of Cattani-Deligne-Kaplan.
- Let $\mathbf{H}$ be a mixed Hodge structure. The *Hodge group* of $\mathbf{H}$ is the group $\mathrm{Hdg}(\mathbf{H}) := \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(0), \mathbf{H})$. We can (and do) view $\mathrm{Hdg}(\mathbf{H})$ as a subgroup of $\mathbf{H}$ via the map $f \mapsto f(1)$.
- If X is a smooth, projective algebraic variety, the Hodge group of $H^{2k}(X, \mathbb{Z}(k))$ the group of Hodge classes on X. The Hodge conjecture says that an non-zero integral multiple of every such class is algebraic.

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- Let $f : X \to S$ be a smooth, projective morphism of algebraic varieties. One obtains, for each integer k, a variation of pure Hodge structure of weight 0, $\mathbf{H} := R^{2k} f_* \mathbb{Z}(k)$. Given a Hodge class in $\omega_{s_0} \in \mathbf{H}_{s_0}$ for $s_0 \in S$, one can translate ω_{s_0} by parallel transport to a class $\omega_s \in \mathbf{H}_s$ and consider the locus Z on S where the class remains Hodge.
- It is easy to see that the locus Z at least if one transports only in a small ball around s_0. This is because the Hodge filtration is a holomorphic subbundle of $\mathbf{H}_{\mathcal{O}_S}$.
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Theorem (CDK)

Let $\mathbf{H}$ *be a pure variation of Hodge structure on a smooth algebraic variety* S *contained as a Zariski open subset of a smooth projective algebraic variety* $\bar{S}$*. Let* $\mathrm{Hdg}(\mathbf{H})$ *denote the locus of Hodge classes in the étalé space of* $\mathbf{H}$*. Then every component of* $\mathrm{Hdg}(\mathbf{H})$ *extends to an analytic space finite and proper over* $\bar{S}$*.*

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Splittings

Let $\mathbf{V} = (V, F, W)$ be a mixed Hodge structure. Deligne showed that there is a unique bigrading $I^{p,q}$ of $\mathbf{V}$ with the following properties

- $F^p = \oplus_{p' > p} I^{p',q}$,
- $W_n = \oplus_{p+q \leq n} I^{p,q}$,
- $I^{q,p} = \bar{I}^{p,q} \mod \oplus_{p'<p, q'<q} I^{p',q'}$.

Define an endomorphism $Y = Y_{(F,W)} \in \operatorname{End} V$ by setting $Y(v) = (p+q)v$ for $v \in I^{p,q}$. Then Y is a splitting of the weight filtration: $W_n = \oplus_{k \leq n} E_k(Y)$.

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- A mixed Hodge structure **V** is split ($\cong \oplus \mathrm{Gr}_k^W \mathbf{V}$) iff $Y \in End V_{\mathbb{Z}}$. This is easy to see because Y preserves the Hodge and weight filtrations.
- So if ν is an admissible normal function given by an extension

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- To prove the theorem we are reduce by GAGA to proving that the closure of $Z(\nu)$ is analytic. We can then replace S with $(\Delta^*)^r$ and $\bar{S}$ with Δ^r.
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Theorem (B-Pearlstein)

Suppose $r = 1$ and $\mathbf{H}$ has weight -1. Then $\lim_{s\to 0} Y_{(F(s),W)}$ exists along any angular sector.

- This is the main tool used to prove the theorem under the above hypotheses. Obviously the zero locus does not have an accumulation point at 0 if the limit is non-integral.
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sl_2 splitting

- Let $\mathbf{V} = (V, W, F)$ be a mixed Hodge structure.
- We say that V is *split over* $\mathbb{R}$ if $V_\mathbb{R} \cong \bigoplus \mathrm{Gr}_k^W V_\mathbb{R}$ as a real mixed Hodge structure.
- Cattani, Kaplan and Schmid define a canonical operator $\xi \in \mathrm{End}\, V_\mathbb{C}$ such that the $(V, W, e^\xi F)$ is split over $\mathbb{R}$. If $\mathbf{V}$ is already split over $\mathbb{R}$ then $\xi = 0$.
- Write $\hat{F} := e^\xi F$ and $\hat{Y} := Y_{(\hat{F}, W)}$. Then $\hat{Y}$ is a real splitting of W called the sl_2-splitting.
- Hodge structure that are split over $\mathbb{R}$ are extremely important in the asymptotics of variations of Hodge structure. Unfortunately, while ξ is canonical it is not unique and it is not the easiest way to split $\mathbf{V}$ over $\mathbb{R}$.

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Deligne's splittings

Let (V, W, N) be a triple consisting of a vector space, an exhaustive increasing filtration and a nilpotent operator preserving the filtration. Suppose that the relative weight filtration $M = M(N, W)$ exists. Let Y_M be a splitting of M. We can find splittings Y_W commuting with Y_M. Given any such, we can use it to grade the elements of End V. So, for example, write $N = \sum_{k \leq 0} N_k$ where $N_k \in EndV$.

Theorem (Deligne)

There exists a unique grading $Y_W = Y(N, Y_M)$ of W commuting with Y_M such that $(N_0, Y_W - Y_M)$ produces and sl_2 triple and, for each $k < 0$, N_k is a highest weight vector.

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Let

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be an admissible normal function over Δ^* *with* $\mathbf{H}$ *of weight* -1. *Suppose* $\mathbf{V}$ *has local normal form*

$$F(z) = e^{zN} e^{\Gamma(s)} F_{\infty}.$$

Then $\lim_{s \to 0} Y_{(F,W)} = Y(N, Y_{(\hat{F},M)})$ *along any angular sector.*

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Then $Y_{(\hat{F}(z),W)}$ is bounded on on U^r.

The proof proceeds by computing the limit of a sequence along certain subsequences where the imaginary parts of the z_i tend to infinity in a prescribed way. The main tool is the sl_2 orbit theorem of Kato-Nakayama-Usui.

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