Feynman diagrams and their periods

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Outline

MZVs

- $\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$ with $s \in \mathbb{Z}_{>1}$ are special values of zeta function.
- If $(s_1, \ldots, s_k) \in \mathbb{Z}_{>0}$ with $s_k > 1$, then

$$\zeta(s_1, \ldots, s_k) := \sum_{0<n_1<\ldots<n_k<1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

  is multiple zeta value (MZV).
- These can be represented as period integrals

$$\zeta(s) = \int_{0<x_1<\cdots x_s<1} \frac{dx_1 \cdots dx_n}{(1-x_1)x_2 \cdots x_s}$$

  and there is a similar formula for $\zeta(s_1, \ldots, s_k)$.

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Periods

- Here a period integral is a complex number whose real and imaginary parts are of the form

$$\int_D f dx_1 \cdots dx_n$$

with $f \in \mathbb{Q}(x_1, \ldots, x_n)$ and $D \subset \mathbb{R}^n$ defined by polynomial inequalities with rational coefficients.

- $\mathcal{P}$ = set of periods. Obviously countable.
- KZ-Challenge. Find a number in $\mathbb{C} - \mathcal{P}$.

Periods and cohomology

Another way to describe period integrals is as follows:

- X alg. var. over $\overline{\mathbb{Q}}$.
- $\omega \in H^n_{\mathrm{dR}} X$ = cohomology of X calculated with cochains defined over $\overline{\mathbb{Q}}$.
- $\gamma \in H_n(X(\mathbb{C}), \mathbb{Q})$ = homology of $X(\mathbb{C})$ computed with simplices.
- There is a pairing

$$\begin{aligned} H^n_{\mathrm{dR}} \otimes H_n(X(\mathbb{C}), \mathbb{Q}) &\to \mathbb{C} \\ \omega \otimes \gamma &\mapsto \int_\gamma \omega \end{aligned}$$

A period (of X) is a number of the form $\int_\gamma \omega$.

Periods and Motives

- If we have a subvariety $D \subset X$ and $\omega \in \Omega^n X$, $\gamma \in H_n(X, D)$, then $\int_\gamma \omega$ is also a period. ($n = \dim X$).
- All periods are of this form with X affine and smooth, D a divisor.
- If we can find a summand $M \subset H^n(X)$ which is natural with respect to algebraic geometry (a motive), and $\omega \in M$, $\gamma \in M^\vee$, then $\int_\gamma \omega$ is called a period of M.

Examples

- $X = \mathbb{C} - \{0\}, \omega = dz/z, \gamma =$ loop around 0.

$$\int_\gamma \omega = 2\pi i, \quad \text{a period of } \mathbb{C} - \{0\}.$$

- $X = \mathbb{C} - \{0\}, D = \{1, 2\}, \omega = dz/z, \gamma =$ path from 1 to 2.

$$\int_\gamma \omega = \ln 2, \quad \text{a period of } (\mathbb{C}, \{1, 2\}).$$

Structure on Cohomology

- X smooth, projective

$$\Rightarrow H^n(X, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}$$

with $\overline{H^{p,q}} = H^{q,p}$. This is called a **pure Hodge structure** on X. It is very fine grained information.

- X quasi-projective. $H^n(X)$ has a mixed Hodge structure. A successive extension of pure Hodge structures.
- $X/\overline{\mathbb{Q}}$ also has étale cohomology. $H^n_{\text{et}}(X, \mathbb{Q}_l)$ is a $\mathbb{Q}_l$ vector space equipped with an action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Tate objects

The *Tate motive* is the motive $\mathbb{Q}(n) = H_n(\mathbb{P}^n)$ which we understand through its Hodge and étale realizations as follows.

- In Hodge theory, $\mathbb{Q}(n) :=$ the pure Hodge structure H with

$$H^{p,q} = \begin{cases} \mathbb{C} & p = q = -n, \\ 0 & \text{else.} \end{cases}$$

  NB: $H^n(\mathbb{P}^n) = \mathbb{Q}(-n)$.

- In étale cohomology, $\mathbb{Q}_l(n) :=$ the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module equal to $\mathbb{Q}_l$ with $\mathrm{Frob}_p(1) = p^n$, ($p \neq l$).

$$H^n_{\mathrm{et}}(\mathbb{P}^n, \mathbb{Q}_l) = \mathbb{Q}_l(-n).$$

- A motive $M \subset H^n(X)$ is called **mixed Tate** if M is a successive extension of Tate objects.

Indoctrination

- We believe (by the Beilison conjectures) that $\int_\gamma \omega$ is an MZV if and only if there is a mixed Tate M with $\omega \in M, \gamma \in M^\vee$.
- We believe that there is an abelian category of mixed Tate motives over any field. Moreover the mixed Tate motives control K-theory. For example by Deligne-Goncharov,

$$K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Q} = \mathrm{Ext}^1_{MTM}(\mathbb{Q}, \mathbb{Q}(n)).$$

- We know that

$$K_{2n-1} \otimes \mathbb{Q} := \begin{cases} 0 & n \text{ even}, \\ \mathbb{Q} & n \text{ odd}. \end{cases}$$

and $\zeta(n)$ is a period of the nontrivial class in the corresponding Ext-group (Borel's theorem).

Graphs

- G a (di)graph.
- $V = V(G) = \{v_1, \dots, v_n\}$ vertices.
- $E = E(G) \subset V \times V$— directed edges, but direction won't really matter.
- Pick $D \in \mathbb{Z}_{>0}$ and get a chain complex

$$\begin{array}{ccccccc} H_1(G,\mathbb{R}^D) & \longrightarrow & C_1(G,\mathbb{R}^D) & \xrightarrow{\partial} & C_0(G,\mathbb{R}^D) & \longrightarrow & H_0(G,\mathbb{R}^D) \\ & & \| & & \| & & \\ & & \mathbb{R}\{E\}\otimes\mathbb{R}^D & \longrightarrow & \mathbb{R}\{V\}\otimes\mathbb{R}^D & & \end{array}$$

$$e \otimes p \longmapsto (h(e) - t(e)) \otimes p$$

Feynman diagrams

- Pick $m_e \in \mathbb{C}$ for each edge e — called *masses*.
- Define

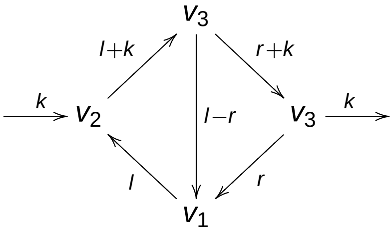
$$\begin{aligned} \Psi : C_1(G, \mathbb{R}^D) &\to \mathbb{C} \\ \Psi(\sum e \otimes p_e) &= \prod (p_e^2 + m_e^2). \end{aligned}$$

- Pick $k \in C_0(G, \mathbb{R}^D)$ — the *external momenta*.
- Set $H_k := \partial^{-1} k$.

$$I(D, k, m) = \int_{H_k} \frac{d\mu}{\Psi(p)}$$ —the *Feynman amplitude*.

Example

The graph G (with only one nonzero mass m_1)



gives the integral $I(D, m, k) =$

$$\int_{\mathbb{R}^{2D}} \frac{d^D l \, d^D r}{((l+k)^2 + m_1^2)(r+k)^2 l^2 r^2 (l-r)^2}$$

Dimensional Regularization

Problem: Does not always converge. Handle this by regularization/renormalization.

- Want to interpolate a function $I(D, m, k)$ meromorphic in D with poles at D when D divergent.
- To simplify $m_e = 1, k = 0$.
- Pick x_e variable for each edge.

We have the *Kirchhoff* polynomial (formed from a sum over trees T in G)

$$P_G := \sum_{T \subset G} \prod_{e \notin T} x_e.$$

$$I_G(D) = I(D, 0, 1) = a(D) \int_{\sum x_e = 1, x_e \geq 0} P_G^{-D/2}$$

$a(D)$ product of Γ-functions & powers of π.

Local Zeta Functions

Theorem (Atiyah, Bernstein-S. Gelfand)

There exists a meromorphic continuation of

$$J(D) = \int P_G^{-D/2}$$

with poles only at $\frac{1}{N}\mathbb{Z}$.

Theorem (Belkale-B.)

Pick $D_0 \in \mathbb{Q}$, *then*

$$J(D) = \sum_{i \geq -N} a_i (D - D_0)^i$$

with a_i *periods.*

Superficial Degree

- Recall our example (for illustration)

$$\int_{\mathbb{R}^{2D}} \frac{d^D l \, d^D r}{((l+k)^2 + m_1^2)(r+k)^2 l^2 r^2 (l-r)^2}.$$

- $\boxed{\delta(G) := b_1 D - 2\#E}$, $b_1 = \#$ loops. Called *Superficial degree of divergence*.
- In our example, $\delta = 2D - 5$.
- $\delta \geq 0 \Rightarrow$ diverges.
- $\delta < 0 \Rightarrow$ converges if denominator non-vanishing and no subgraphs diverge (Weinberg) —called *primitive*.
- $\delta = 0$ logarithmically divergent. Here $I(D)$ should have a simple pole whose residue is independent of m, k.

Example/Conjecture

- G, wheel with n spokes.
- $\delta = nD - 4n = n(D-4) \Rightarrow$ log divergent at $D_0 = 4$.

$$I_G(D) \sim \frac{\zeta(2n-3)}{D-4} + a_0 + \text{higher order.}$$

Conjecture

G primitive, $\delta = 0$. Then

$$I_G(D) = \frac{a_{-1}}{D - D_0} + \textit{higher order}$$

with a_{-1} an MZV.

Conjecture

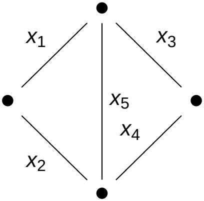
- X a variety with integral coefficients.
- $[X](q) := \#X(\mathbb{F}_q) =$ number of solutions to equations defining X in the field $\mathbb{F}_q$ with q elements.
- G a graph, $X_G :=$ set of solutions to $P_G = 0$.

Conjecture (Kontsevich)

There exists $f \in \mathbb{Z}[t]$ *such that*

$$[X_G](q) = f(q).$$

Example



- $P_G = (x_1 + x_2 + x_3 + x_4)x_5 + (x_1 + x_2)(x_3 + x_4)$.
- $$\begin{aligned}[X_G] &= [\mathbb{A}^4] - [\mathbb{A}^3] + [\mathbb{A}^1][\mathbb{A}^2] \\ &= q^4 - q^3 + q^3 \\ &= q^4.\end{aligned}$$

$\exists$ a Counterexample

Theorem (Belkale-B)

Let X be a variety with integer coefficients. There exists graphs G_i and rational functions $f_i \in \mathbb{Q}(t)$ such that

$$[X] = \sum f_i [X_{G_i}].$$

Corollary

Kontsevich's conjecture is false.

Proof. In emergency, please skip.

Proof.

Take $X = \{x \in \mathbb{A}^1 | 2x = 0\}$.

$$[X] = \begin{cases} q & 2|q, \\ 1 & \text{else.} \end{cases}$$

Clearly not a rational function of q. □

Bloch-Esnault-Kreimer

- G = wheel with n spokes. $D_0 = 4$.
- $a_{-1} \sim \zeta(2n-3)$.
- Should see extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(2n-3)$ in cohomology of $\mathbb{A}^E - X_G$.
- Bloch, Esnault and Kreimer have verified that the $\mathbb{Q}(2n-3)$ term is there.

Summary

- Period integrals are basic objects in algebraic geometry/motives.
- Feynman amplitudes seem to give very interesting periods.
- However, we do not exactly know why or how far this goes.

- Outlook
  - Want an explicit counterexample to Kontsevich.
  - Want to understand cohomology/periods of the X_G when Kontsevich is correct.