Mahler’s measure and L-functions of elliptic curves at $s = 3$

David Boyd

University of British Columbia

SFU-UBC Number Theory Seminar
If \( P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), the **logarithmic Mahler measure** is

\[
m(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}
\]

where \( \mathbb{T}^n = \{ |x_1| = 1 \} \times \cdots \times \{ |x_n| = 1 \} \).

If \( P(x) = a_0 \prod_{j=1}^{d} (x - \alpha_j) \), Jensen gives

\[
m(P) = \log |a_0| + \sum_{j=1}^{d} \log^+ |\alpha_j|,
\]

the logarithm of an algebraic integer if \( P \in \mathbb{Z}[x] \).
Why the torus?

- B-Lawton (1980’s)
  \[ m(P(x, x^{k_2}, \ldots, x^{k_n})) \rightarrow m(P(x_1, \ldots, x_n)), \]
  if \( k_2 \rightarrow \infty, \ldots, k_n \rightarrow \infty \) in a suitable manner.

- so every \( m(P(x_1, \ldots, x_n)) \) is the limit of measures of one-variable polynomials

- This would not be true if we integrated over the \( N \)-ball, for example
Examples with more variables

- Smyth (1981)

\[ m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \]

\[ m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \]

- Notation for some basic constants

\[ d_f = L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2), \quad z_3 = \frac{1}{\pi^2} \zeta(3) \]
Dirichlet L-functions

\[ L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \]

where \( \chi(n) \) is a Dirichlet character

e.g.

\[ L(\chi_{-3}, 2) = 1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \ldots, \]

the signs are given by the Legendre symbol \( \left( \frac{n}{3} \right) \)
Deninger’s insight

- Deninger (1995): Provided $P(x_1, \ldots, x_n) \neq 0$ on $\mathbb{T}^n$, $m(P)$ is related to the cohomology of the variety $\mathcal{V} = \{P(x_1, \ldots, x_n) = 0\}$.

- In particular (here $P = 0$ does intersect $\mathbb{T}^2$ but harmlessly.)

$$m(1 + x + 1/x + y + 1/y) \overset{?}{=} L'(E_{15}, 0)$$

$E_{15}$ the elliptic curve of conductor $N = 15$ defined by $P = 0$.

- More notation

$$b_N = L'(E_N, 0) = \frac{N}{\pi^2} L(E_N, 2),$$

$E_N$ an elliptic curve of conductor $N$. 
Elliptic curve L-functions

\[ L(E, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p} \left( 1 - a_p p^{-s} + p^{1-2s} \right)^{-1} \]

where the \( a_p \) are given by counting points on \( E(\mathbb{F}_p) \)

The \( a_n \) are also the coefficients of a cusp form of weight 2 on \( \Gamma_0(N) \), e.g. for \( N = 15 \),

\[ \sum_{n=1}^{\infty} a_n q^n = q \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{3n})(1 - q^{5n})(1 - q^{15n}) \]
Recipe for making conjectures

- Compute $m(P)$ for lots of $P(x, y)$
- Stir in various constants, $d_3, d_4, d_7, \ldots, b_{11}, b_{14}, \ldots$
- Apply the Lenstra–Lenstra–Lovasz algorithm (LLL)
- Publish the results
Conjectures follow from deeper conjectures

- (B, 1996)

\[ m(k + x + 1/x + y + 1/y) = r_k L'(E_{N_k}, 0), \text{ for specific rationals } r_k \] (\(\star\))

- (Rodriguez-Villegas, 1997)

\[ m(k + x + 1/x + y + 1/y) = \text{Re}( -\pi i \tau + 2 \sum_{n=1}^{\infty} \sum_{d \mid n} \left( \frac{-4}{d} \right) d^2 \frac{q^n}{n} ) \]

where \( q = \exp(\pi i \tau) \) is the modulus of the curve \( E_{N_k} \)

Hence the conjecture (\(\star\)) follows (with generic rationals) from the Bloch-Beilinson conjectures.
Conjectures become Theorems

- (Rodriguez-Villegas, 1997) \((\star)\) is true for \(k = 3\sqrt{2}\)

- (Lalín – Rogers, 2006) \((\star)\) is true for \(k = 2\) and \(k = 8\)

- (Bruneault, 2005)

\[
m(y^2 + (x^2 + 2x - 1)y + x^3) = \frac{5}{4} b_{11},
\]

as conjectured in (B, 1996)
Precursors of a general idea

- (Smyth, 2002)

\[ m(1 + x^{-1} + y + (1 + x + y)z) = \frac{14}{3} \frac{\zeta(3)}{\pi^2} = \frac{14}{3} z_3 \]

- (Lalín, 2003)

\[ m((1 + x_1)(1 + x_2)(1 + x_3) + (1 - x_2)(1 - x_3)(1 + x_4)x_5) = 93 \frac{\zeta(5)}{\pi^4} \]
Maillot’s insight

- (Darboux, 1875) If $P^*(x) = x^{\deg(P)} P(x^{-1})$ is the reciprocal of $P$ and if

$$\mathcal{V} = \{P = 0\}, \quad \text{and} \quad \mathcal{W} = \{P = 0\} \cap \{P^* = 0\},$$

then

$$\mathcal{V} \cap \mathbb{T}^n = \mathcal{W} \cap \mathbb{T}^n$$

- (Maillot, 2003) In case $\mathcal{V}$ intersects $\mathbb{T}^n$ non-trivially, $m(P)$ is related to the cohomology of $\mathcal{W}$.

- The reason that Smyth’s and Lalín’s examples involve only ordinary polylogarithms can be “explained” using this observation.
Elliptic curves again

(Rodriguez-Villegas, 2003) If \( P = (1 + x)(1 + y) + z \), then \( \mathcal{W} \) is an elliptic curve of conductor 15, so perhaps

\[
m(P) \equiv rL'(E_{15}, -1), \quad \text{with} \quad r \in \mathbb{Q}
\]

\[
L'(E_N, -1) = 2\frac{N^2}{(2\pi)^4}L(E_N, 3)
\]

(B, 2003) Yes!

\[
m(P) = 2L'(E_{15}, -1), \text{ to 28 decimal places}
\]

More Notation

\[
L_N = L'(E_N, -1) = 2\frac{N^2}{(2\pi)^4}L(E_N, 3)
\]
Recipe for making more conjectures

- Compute $m(P)$ for lots of $P(x, y, z)$ with $\mathcal{W}$ a curve of small genus
- Stir in various constants, $d_3, d_4, d_7, \ldots, z_3, L_{11}, L_{14}, \ldots$
- Apply the Lenstra–Lenstra–Lovasz algorithm (LLL)
- Show the results to Fernando, Matilde and Mat to see if they can prove them
- Publish the results
A very useful formula

- (Cassaigne–Maillot, 2000) for $a, b, c \in \mathbb{C}^*$,

$$m(a + by + cz) = \begin{cases} \frac{1}{\pi} \left( D\left(\frac{|a|}{|b|}e^{i\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| \right), & \text{if } \triangle \\ \max\{\log |a|, \log |b|, \log |c|\}, & \text{if not } \triangle \end{cases}$$

- The condition $\triangle$ means that $|a|, |b|, |c|$ form the sides of a triangle with angles $\alpha, \beta, \gamma$.

- Bloch–Wigner dilogarithm

$$D(x) := \text{Im}(\text{Li}_2(x)) + \arg(1 - x) \log |x|$$
Computing $m(P)$ for some 3-variable examples

- If $P(x, y, z) = a(x) + b(x)y + c(x)z$ then $f(t) = m(P(e^{it}, y, z))$ is given by the Cassaigne–Maillot formula.

- Numerically integrate to compute

$$m(P(x, y, z)) = \frac{1}{\pi} \int_0^\pi f(t) \, dt$$

- On non-$\triangle$ intervals, we integrate logs and hence obtain dilogs of algebraic numbers.

- On $\triangle$ intervals, we integrate dilogs and hence expect to obtain tri-logs perhaps *elliptic trilogs* ($\longrightarrow L(E, 3)$ by Zagier’s conjecture).
Computing $m(P)$ – a simple example

- $P(x, y, z) = (1 + x)(1 + y) + z$, so $a = b = 1 + x$ and $c = 1$.

- If $x = e^{it}$ then $|a| = |b| = 2 \cos\left(\frac{t}{2}\right)$

- If $|a| > \frac{1}{2}$, i.e. $t < t_0 = 2 \cos^{-1}\left(\frac{1}{4}\right) = 2.63623\ldots$ then we are in the (isoceles) $\triangle$ case with $\gamma = 2 \sin^{-1}\left(\frac{1}{4 \cos(t/2)}\right)$ and $f(t) =$ the hard part of the C-M formula.

- If $t \geq t_0$ then $f(t) = \log |c| = 0$, by the easy part of the formula.

- $m(P) = 0.4839979734786385357732733911 = 2L_{15}$ to 28 d.p.
Which polynomials?

- \( P(x, y, z) = a(x) + b(x)y + c(x)z \), with \( a, b, c \) cyclotomic of degree \( \leq 4 \)

- Eliminate \( z \) from \( P \) and \( P^* \) to obtain an equation for
  \[ \mathcal{W} = \{ Q(x, y) = 0 \} \]

- \( \mathcal{W} \) is the hyperelliptic curve \( Y^2 = \text{disc}_y Q \), genus \( g \), say

- If \( g \leq 2 \) compute \( m(P) \) and apply the recipe of the previous slide to make conjectures
Examples of genus 0


   \[ m(x - 1 + (x + 1)(y + z)) = \frac{28}{5}z_3 \]

2.

   \[ m(x^2 + 1 + (x^2 + x + 1)(y + z)) = \frac{10}{9}d_3 + \frac{35}{18}z_3 \]

   Matilde Lalín and Mat Rogers each have proofs of this (2006)

3.

   \[ m((x - 1)^2 + (x^2 + 1)(y + z)) = -d_3 + 2d_4 \]

   NB: no trilog term here – we do integrate a dilog. The negative coefficient of \( d_3 \) is also notable (and useful).
Examples of genus 1

1. A mixture of a dilog and \(L(E_{45}, 3)\)

\[
m(1 + (x^2 - x + 1)y + (x^2 + x + 1)z) \equiv d_3 + \frac{1}{6}L_{45}
\]

2. Here \(\mathcal{W}\) is an elliptic curve of conductor 57, but \(m(P)\) is an ordinary trilog.

\[
m(x^2 + x + 1 + (x^2 - 1)(y + z)) \equiv \frac{28}{5}z_3
\]

3. Here we have both the ordinary trilog \(z_3\) and an elliptic trilog \(L_{21}\)

\[
m(x - 1 + (x^2 - 1)y + (x^2 + x + 1)z) \equiv \frac{2}{3}d_3 + \frac{199}{72}z_3 + \frac{11}{24}L_{21}
\]
Examples of genus 2

- Since $Q(x, y)$ is reciprocal $\text{Jac}(\mathcal{W}) = E \times F$ for elliptic curves $E, F$ (Jacobi, 1832)

1. Here $E$ and $F$ have conductors 14 and $112 = 2^4 \cdot 7$

   $$m((x - 1)^3 + (x + 1)(y + z)) \equiv 6L_{14}$$

2. Here $E$ and $F$ have conductors 108 and 36 ($E : Y^2 = X^3 + 4$)

   $$m((x - 1)^3 + (x + 1)^3(y + z)) \equiv -\frac{28}{15}z_3 + \frac{2}{15}L_{108}$$
Representing $L(E, 3)$

- Combining the genus 0 example

  $$m(P_1) = m(x - 1 + (x + 1)(y + z)) = \frac{28}{5} z_3$$

- with the genus 2 example

  $$m(P_2) = m((x - 1)^3 + (x + 1)^3(y + z)) = -\frac{28}{15} z_3 + \frac{2}{15} L_{108}$$

- we obtain

  $$m(P_1 P_2^3) = \frac{2}{5} L_{108},$$

  showing the importance of negative coefficients in these formulas


1. This example is of genus 1 with $E : Y^2 = X^3 + X$ of conductor 64, but we have no formula for $m(P_1)$

$$P_1 = (x - 1)^2 + (x + 1)^2(y + z)$$

We also have no formula for the following genus 2 example with $\text{Jac} = E \times F$ with $E, F$ of conductors 64, 192

$$P_2 = (x + 1)^2 + (x^4 + 1)y + (x^2 + 1)(x^2 + x + 1)z$$

However, the missing ingredients of the two formulas must be the same since

$$m(P_1^{12}P_2^{19}) = 12m(P_1) + 19m(P_2) \equiv -19d_3 + 20d_4 + 6L_{64}$$