Mahler’s measure and $L$-functions of elliptic curves at $s = 3$

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SFU-UBC Number Theory Seminar
If $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the \textit{logarithmic Mahler measure} is

$$m(P) = \frac{1}{(2\pi i)^n} \int_{T^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

where

$$T^n = \{|x_1| = 1\} \times \cdots \times \{|x_n| = 1\}$$

If $P(x) = a_0 \prod_{j=1}^{d} (x - \alpha_j)$, Jensen gives

$$m(P) = \log |a_0| + \sum_{j=1}^{d} \log^+ |\alpha_j|,$$

the logarithm of an algebraic integer if $P \in \mathbb{Z}[x]$. 

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Mahler measure and $L(E, 3)$

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Why the torus?

- B-Lawton (1980’s)

\[ m(P(x, x^{k_2}, \ldots, x^{k_n})) \rightarrow m(P(x_1, \ldots, x_n)), \]

if \( k_2 \rightarrow \infty, \ldots, k_n \rightarrow \infty \) in a suitable manner.

- so every \( m(P(x_1, \ldots, x_n)) \) is the limit of measures of one-variable polynomials

- This would not be true if we integrated over the \( N \)-ball, for example
Examples with more variables

- Smyth (1981)

\[ m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \]

\[ m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \]

- Notation for some basic constants

\[ d_f = L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2), \quad z_3 = \frac{1}{\pi^2} \zeta(3) \]
Dirichlet L-functions

\[ L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \]

where \( \chi(n) \) is a Dirichlet character

e.g.

\[ L(\chi_{-3}, 2) = 1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \ldots, \]

the signs are given by the Legendre symbol \( \left( \frac{n}{3} \right) \)
Deninger’s insight

- Deninger (1995): Provided $P(x_1, \ldots, x_n) \neq 0$ on $\mathbb{T}^n$, $m(P)$ is related to the cohomology of the variety $\mathcal{V} = \{ P(x_1, \ldots, x_n) = 0 \}$.

- In particular (here $P = 0$ does intersect $\mathbb{T}^2$ but harmlessly.)

\[ m(1 + x + 1/x + y + 1/y) \equiv L'(E_{15}, 0) \]

$E_{15}$ the elliptic curve of conductor $N = 15$ defined by $P = 0$.

- More notation

\[ b_N = L'(E_N, 0) = \frac{N}{\pi^2} L(E_N, 2), \]

$E_N$ an elliptic curve of conductor $N$. 
Elliptic curve L-functions

\[ L(E, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p} \left( 1 - a_p p^{-s} + p^{1-2s} \right)^{-1} \]

where the \( a_p \) are given by counting points on \( E(\mathbb{F}_p) \)

The \( a_n \) are also the coefficients of a cusp form of weight 2 on \( \Gamma_0(N) \), e.g. for \( N = 15 \),

\[ \sum_{n=1}^{\infty} a_n q^n = q \prod_{n=1}^{\infty} \left( 1 - q^n \right) \left( 1 - q^{3n} \right) \left( 1 - q^{5n} \right) \left( 1 - q^{15n} \right) \]
Recipe for making conjectures

- Compute $m(P)$ for lots of $P(x, y)$
- Stir in various constants, $d_3, d_4, d_7, \ldots, b_{11}, b_{14}, \ldots$
- Apply the Lenstra–Lenstra–Lovasz algorithm (LLL)
- Publish the results
Conjectures follow from deeper conjectures

- (B, 1996)

\[ m(k + x + 1/x + y + 1/y) \overset{?}{=} r_k L'(E_{N_k}, 0), \text{ for specific rationals } r_k \] 

- (Rodriguez-Villegas, 1997)

\[ m(k + x + 1/x + y + 1/y) = \Re(-\pi i \tau + 2 \sum_{n=1}^{\infty} \sum_{d|n} \left( -\frac{4}{d} \right) d^2 \frac{q^n}{n}) \]

where \( q = \exp(\pi i \tau) \) is the modulus of the curve \( E_{N_k} \)

- Hence the conjecture \((\ast)\) follows (with generic rationals) from the Bloch-Beilinson conjectures.
Conjectures become Theorems

- (Rodriguez-Villegas, 1997) (*) is true for $k = 3\sqrt{2}$
- (Lalín – Rogers, 2006) (*) is true for $k = 2$ and $k = 8$
- (Brunault, 2005)

$$m(y^2 + (x^2 + 2x - 1)y + x^3) = \frac{5}{4} b_{11},$$

as conjectured in (B, 1996)
Precursors of a general idea

- (Smyth, 2002)
  \[ m(1 + x^{-1} + y + (1 + x + y)z) = \frac{14}{3} \frac{\zeta(3)}{\pi^2} = \frac{14}{3} z_3 \]

- (Lalín, 2003)
  \[ m((1 + x_1)(1 + x_2)(1 + x_3) + (1 - x_2)(1 - x_3)(1 + x_4)x_5) = 93 \frac{\zeta(5)}{\pi^4} \]
Maillot’s insight

(Darboux, 1875) If $P^*(x) = x^{\deg(P)} P(x^{-1})$ is the reciprocal of $P$ and if

$$V = \{ P = 0 \}, \quad \text{and} \quad W = \{ P = 0 \} \cap \{ P^* = 0 \},$$

then

$$V \cap \mathbb{T}^n = W \cap \mathbb{T}^n$$

(Maillot, 2003) In case $V$ intersects $\mathbb{T}^n$ non-trivially, $m(P)$ is related to the cohomology of $W$.

The reason that Smyth’s and Lalín’s examples involve only ordinary polylogarithms can be “explained” using this observation.
Elliptic curves again

(Rodriguez-Villegas, 2003) If \( P = (1 + x)(1 + y) + z \), then \( \mathcal{W} \) is an elliptic curve of conductor 15, so perhaps

\[
m(P) \equiv rL'(E_{15}, -1), \quad \text{with} \quad r \in \mathbb{Q}
\]

\[
L'(E_{N}, -1) = 2 \frac{N^2}{(2\pi)^4} L(E_{N}, 3)
\]

(B, 2003) Yes!

\[
m(P) = 2L'(E_{15}, -1), \text{ to 28 decimal places}
\]

More Notation

\[
L_{N} = L'(E_{N}, -1) = 2 \frac{N^2}{(2\pi)^4} L(E_{N}, 3)
\]
Recipe for making more conjectures

- Compute $m(P)$ for lots of $P(x, y, z)$ with $\mathcal{W}$ a curve of small genus
- Stir in various constants, $d_3, d_4, d_7, \ldots, z_3, L_{11}, L_{14}, \ldots$
- Apply the Lenstra–Lenstra–Lovász algorithm (LLL)
- Show the results to Fernando, Matilde and Mat to see if they can prove them
- Publish the results
A very useful formula

- (Cassaigne–Maillot, 2000) for $a, b, c \in \mathbb{C}^*$,

$$m(a + by + cz) = \begin{cases} \frac{1}{\pi} \left( \mathcal{D}\left( \frac{|a|}{|b|} e^{i\gamma} \right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| \right), & \text{if } \triangle \\ \max\{\log |a|, \log |b|, \log |c|\}, & \text{if not } \triangle \end{cases}$$

- The condition $\triangle$ means that $|a|, |b|, |c|$ form the sides of a triangle with angles $\alpha, \beta, \gamma$.

- Bloch–Wigner dilogarithm

$$\mathcal{D}(x) := \text{Im}(\text{Li}_2(x)) + \text{arg}(1 - x) \log |x|$$
Computing $m(P)$ for some 3-variable examples

- If $P(x, y, z) = a(x) + b(x)y + c(x)z$ then $f(t) = m(P(e^{it}, y, z))$ is given by the Cassaigne–Maillot formula.

- Numerically integrate to compute

$$m(P(x, y, z)) = \frac{1}{\pi} \int_{0}^{\pi} f(t) \, dt$$

- On non-$\triangle$ intervals, we integrate logs and hence obtain dilogs of algebraic numbers.

- On $\triangle$ intervals, we integrate dilogs and hence expect to obtain *tri-logs* perhaps *elliptic trilogs* ($\rightarrow L(E, 3)$ by Zagier’s conjecture).
Computing $m(P)$ – a simple example

- $P(x, y, z) = (1 + x)(1 + y) + z$, so $a = b = 1 + x$ and $c = 1$.

- If $x = e^{it}$ then $|a| = |b| = 2 \cos \left( \frac{t}{2} \right)$

- If $|a| > \frac{1}{2}$, i.e. $t < t_0 = 2 \cos^{-1} \left( \frac{1}{4} \right) = 2.63623 \ldots$ then we are in the (isoceles) $\Delta$ case with $\gamma = 2 \sin^{-1} \left( \frac{1}{4 \cos(t/2)} \right)$ and $f(t) =$ the hard part of the C-M formula.

- If $t \geq t_0$ then $f(t) = \log |c| = 0$, by the easy part of the formula.

- $m(P) = 0.4839979734786385357732733911 = 2L_{15}$ to 28 d.p.
Which polynomials?

\[ P(x, y, z) = a(x) + b(x)y + c(x)z, \text{ with } a, b, c \text{ cyclotomic of degree } \leq 4 \]

- Eliminate \( z \) from \( P \) and \( P^* \) to obtain an equation for

\[ \mathcal{W} = \{ Q(x, y) = 0 \} \]

- \( \mathcal{W} \) is the hyperelliptic curve \( Y^2 = \text{disc}_y Q \), genus \( g \), say

- If \( g \leq 2 \) compute \( m(P) \) and apply the recipe of the previous slide to make conjectures
Examples of genus 0


\[ m(x - 1 + (x + 1)(y + z)) = \frac{28}{5}z_3 \]

2. 

\[ m(x^2 + 1 + (x^2 + x + 1)(y + z)) = \frac{10}{9}d_3 + \frac{35}{18}z_3 \]

Matilde Lalín and Mat Rogers each have proofs of this (2006)

3. 

\[ m((x - 1)^2 + (x^2 + 1)(y + z)) = -d_3 + 2d_4 \]

NB: no trilog term here – we do integrate a dilog. The negative coefficient of \( d_3 \) is also notable (and useful).
Examples of genus 1

1. A mixture of a dilog and $L(E_{45}, 3)$

$$m(1 + (x^2 - x + 1)y + (x^2 + x + 1)z) = d_3 + \frac{1}{6}L_{45}$$

2. Here $\mathcal{W}$ is an elliptic curve of conductor 57, but $m(P)$ is an ordinary trilog.

$$m(x^2 + x + 1 + (x^2 - 1)(y + z)) = \frac{28}{5}z_3$$

3. Here we have both the ordinary trilog $z_3$ and an elliptic trilog $L_{21}$

$$m(x - 1 + (x^2 - 1)y + (x^2 + x + 1)z) = \frac{2}{3}d_3 + \frac{199}{72}z_3 + \frac{11}{24}L_{21}$$
Examples of genus 2

- Since $Q(x, y)$ is reciprocal, $\text{Jac}(\mathcal{W}) = E \times F$ for elliptic curves $E, F$ (Jacobi, 1832)

1. Here $E$ and $F$ have conductors 14 and $112 = 2^4 \cdot 7$:

\[ m((x - 1)^3 + (x + 1)(y + z)) \approx 6L_{14} \]

2. Here $E$ and $F$ have conductors 108 and 36 ($E : Y^2 = X^3 + 4$):

\[ m((x - 1)^3 + (x + 1)^3(y + z)) \approx -\frac{28}{15}z_3 + \frac{2}{15}L_{108} \]
Representing $L(E, 3)$

- Combining the genus 0 example

\[ m(P_1) = m(x - 1 + (x + 1)(y + z)) = \frac{28}{5} z_3 \]

- with the genus 2 example

\[ m(P_2) = m((x - 1)^3 + (x + 1)^3(y + z)) = -\frac{28}{15} z_3 + \frac{2}{15} L_{108} \]

- we obtain

\[ m(P_1 P_2^3) = \frac{2}{5} L_{108}, \]

showing the importance of negative coefficients in these formulas.
Exotic formulas

1. This example is of genus 1 with $E : Y^2 = X^3 + X$ of conductor 64, but we have no formula for $m(P_1)$

$$P_1 = (x - 1)^2 + (x + 1)^2(y + z)$$

We also have no formula for the following genus 2 example with $\text{Jac} = E \times F$ with $E, F$ of conductors 64, 192

$$P_2 = (x + 1)^2 + (x^4 + 1)y + (x^2 + 1)(x^2 + x + 1)z$$

However, the missing ingredients of the two formulas must be the same since

$$m(P_{12}^{12}P_{19}^{19}) = 12m(P_1) + 19m(P_2) \equiv -19d_3 + 20d_4 + 6L_{64}$$