Mathematics 539 – Exercises # 1
Arithmetical Functions

1. Prove that
\[ \frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}. \]

2. Prove that
\[ \sum_{d^2|n} \mu(d) = \mu^2(n) \]
and, more generally,
\[ \sum_{d^k|n} \mu(d) = \begin{cases} 0 & \text{if } m^k|n \text{ for some } m > 1, \\ 1 & \text{otherwise}. \end{cases} \]

3. Suppose that \( f(x) \) is defined for \( 0 \leq x \leq 1 \) and we define
\[ F(n) = \sum_{1 \leq k \leq n} f(k/n) \quad \text{and} \quad G(n) = \sum_{1 \leq k \leq n} f(k/n). \]
(a) Prove that \( G(n) = \sum_{d|n} \mu(d) F(n/d). \)
(b) Prove that \( \mu(n) \) is the sum of the primitive \( n \)th roots of unity:
\[ \mu(n) = \sum_{1 \leq k \leq n \atop (k,n)=1} \exp(2\pi i k/n). \]

4. Let \( f(n) = [\sqrt{n}] - [\sqrt{n-1}] \). Show that \( f \) is multiplicative but not completely multiplicative.

5. Prove or disprove the following: if \( f(n) \) is multiplicative, and if \( F(n) = \prod_{d|n} f(d), \) then \( F(n) \) is multiplicative.

6. Prove or disprove the following:
(a) If \( f(n) \) is a multiplicative function for which \( f(p^m) \to 0 \) as \( m \to \infty \) for each prime \( p \), then \( f(n) \to 0 \) as \( n \to \infty \).
(b) If \( f(n) \) is multiplicative and \( f(n) \to 0 \) as \( n \to \infty \) through the sequence of all prime powers \( \{p^m\} = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, \ldots \} \) then \( f(n) \to 0 \) as \( n \to \infty \). More explicitly, the hypothesis of (b) means that if we arrange the prime powers in a sequence \( q_1 < q_2 < \ldots \) then \( \lim_{k \to \infty} f(q_k) = 0 \).

7. Let \( \lambda(n) \) denote Liouville’s function.
(a) Prove that
\[ \sum_{d|n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square}, \\ 0, & \text{otherwise}. \end{cases} \]
(b) Prove that
\[ \lambda(n) = \sum_{m^2|n} \mu(n/m^2). \]