**Mathematics 437/537 – Homework #6**

due Wednesday, November 29, 2006 at the beginning of class

1. (NZM 9.5 – 7 (H)) Prove that in any quadratic field $\mathbb{Q}(\sqrt{d})$ the following assertion is false: If $N(\alpha)$ is a rational integer then $\alpha$ is an algebraic integer.

2. (NZM 9.7 – 3) Prove that $11 + 2\sqrt{6}$ is a prime in $\mathbb{Q}(\sqrt{6})$.

3. (NZM 9.7 – 5) Prove that there are infinitely many primes in any quadratic field $\mathbb{Q}(\sqrt{d})$.

4. Prove that $Q(\sqrt{d})$ is a Euclidean field if and only if for every $x + y\sqrt{d} \in Q(\sqrt{d})$, there is an algebraic integer $a + b\sqrt{d} \in Q(\sqrt{d})$ such that $N(x + y\sqrt{d} - a - b\sqrt{d}) < 1$.

5. (NZM 9.8 – 1) Prove that $\mathbb{Q}(\sqrt{-11})$ is a Euclidean field.

6. Prove that $\mathbb{Q}(\sqrt{13})$ is a Euclidean field.

7. (NZM 9.9 – 5) Prove that the primes of $\mathbb{Q}(\sqrt{2})$ are $\sqrt{2}$, all rational primes of the form $8k + 3$, all factors $a + b\sqrt{2}$ of rational primes of the form $8k \pm 1$, and all associates of these primes.

8. (NZM 9.9 –*6 (H)) Prove that if $d$ is square-free, $d < -1$, $|d|$ is not a prime, then $\mathbb{Q}(\sqrt{d})$ does not have the unique factorization property.

9. (NZM 9.9 – 7) Find all solutions of $y^2 + 1 = x^3$ in rational integers

10. (similar to NZM 5.4 – 11) Define $f(x) = (x^2 - 2)(x^2 - 17)(x^2 - 34)$. Prove that for every integer $m$, the congruence $f(x) \equiv 0 \pmod{m}$ has a solution.