Mathematics 437/537 – Homework #5

due Wednesday, November 15, 2006 at the beginning of class

1. (NZM 3.6 – 8) Prove that if a positive integer n can be expressed as a sum of the squares of two rational numbers then it can be expressed as a sum of the squares of two integers.

2. (based on NZM 3.6–11,12) Show that if n is of the form $4^m(8k+7)$ for non-negative integers k, m, then there are no integers x, y, z for which $n = x^2 + y^2 + z^2$.

3. Let c, d be integers with $d > 0$. Define $u_0 = c$, $u_1 = d$ and use the notation of (7.1) on NZM p.325 for the Euclidean algorithm (but assume $(c, d) = g$ where g need not be 1). Let x_n, y_n be the integers determined by the Extended Euclidean algorithm, such that $u_n = x_n c + y_n d$ for $n = 0, \ldots, j + 1$. Determine (and prove) the relationship between x_n, y_n and the p_n, q_n for which $\langle a_0, \ldots, a_n\rangle = p_n/q_n$ with $(p_n, q_n) = 1$.

4. (NZM 7.3 – 6) Given a prime number p with $p \equiv 1 \pmod 4$ and a $u > 0$ with $u^2 \equiv -1$, let $u/p = \langle a_0, \ldots, a_n\rangle$ and let k be the largest integer such that $q_k \leq \sqrt{p}$. Put $x = q_k$ and $y = |p_k p - q_k u|$. Show that $y \leq \sqrt{p}$, that $0 < x^2 + y^2 < 2p$ and that $x^2 + y^2 \equiv 0 \pmod p$, and hence that $x^2 + y^2 = p$. (This is a method due to Hermite, 1848).

5. (NZM 7.4 - *6) Let $\theta = \langle a_0, a_1, \ldots\rangle$, and as usual (for us) write $p_n/q_n = \langle a_0, \ldots, a_n\rangle$ and $\theta_n = \langle a_n, a_{n+1}, \ldots\rangle$. Show that for $n \geq 1$,
$$\theta - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n^2(\theta_{n+1} + \langle 0, a_n, a_{n-1}, \ldots, a_2, a_1\rangle)}.$$

6. Of the first 1000 multiples of e, which one is closest to an integer? (Please don't do this by brute force!)

7. (NZM 7.8 - 10) In this problem, "smallest solution" means the ordered pair (x, y) of positive integers with y as small as possible.
   (a) Compute the continued fraction expansion of $\sqrt{61}$.
   (b) Find the smallest solution to $x^2 - 61y^2 = -4$.
   (c) Find the smallest solution to $x^2 - 61y^2 = -1$.
   (d) Find the smallest solution to $x^2 - 61y^2 = 1$.
   (Hint: What is the relationship between the solutions for (b),(c) & (d)?)

8. Suppose that p is a prime with $p \equiv 1 \pmod 4$. Show that $x^2 - py^2 = -1$ has an integral solution. (NZM 7.8 - 12 outlines a proof).

9. (a) Given an integer $k \geq 2$, compute the continued fraction expansions of $\sqrt{k^2 + 4}$.
   (b) Prove that the equation $x^2 + 1 = (k^2 + 4)y^2$ has no integer solution (x, y) if k is even, but infinitely many integer solutions if k is odd.

10. Show that $x^2 - 34y^2 = -1$ has no integral solution but that it does have the rational solutions $(x, y) = (5/3, 1/3)$ and $(3/5, 1/5)$. Use these to show that $x^2 - 34y^2 \equiv -1 \pmod m$ has a solution for all positive integers m. (NZM 7.8 - 13 has some hints).