Mathematics 437/537 – Homework #1

due Monday, September 18, 2006 at the beginning of class

Marking: When marking the homework I'll consider the clarity of your explanations as well as their correctness. Attempt all the problems and hand in those that you solve. You may discuss problems among yourselves but the solutions you submit should be written by you in your own words.

1. Calculate $(2310, 649)$. Find integers x and y such that $2310x + 649y = (2310, 649)$. (Do this problem by hand, showing your work).

2. For this problem, use only material up to Section 1.2 in your proofs. (In particular, don't use the notion of a prime).
   (a) Suppose that a, b and d are integers with $d|ab$. Prove that there are integers e and f with $e|a$ and $f|b$ such that $d = ef$.
   (b) If we add the assumption in (a) that $(a, b) = 1$, prove that the integers e and f are unique up to sign. Show by example that without the added assumption, this uniqueness can fail.

3. (NZM 1.3 – 26) Show that there are infinitely many primes of the form $4n+3$; of the form $6n + 5$.

4. (NZM 1.3 – *36) Consider the set S of integers $\{1, 2, \ldots, n\}$. Let 2^k be the highest power of 2 in S. Show that 2^k does not divide any other element of S. Use this to show that $\sum_{j=1}^{n} 1/j$ is not an integer for any $n > 1$.

5. Define $K(i, j)$ to be the number with the decimal expansion: $K(i, j) = \underbrace{11 \ldots 11}_{i}\underbrace{00 \ldots 00}_{j}$.
   Prove that any positive integer n divides some positive integer of the form $K(i, j)$.

6. Find the last three digits of $987^{(1203^{321})}$. (Hint: Euler's theorem – no calculator needed! You may use the formula for $\phi(n)$ of Thm 2.19, p.69.)

7. (NZM 2.1 – 20). Prove that $n^7 - n$ is divisible by 42, for any integer n.

8. (NZM 2.1 – 27). Prove that $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is an integer for every integer n.

9. Let p be an odd prime and r a positive integer. Let $k = \phi(p^r)$ and let $\{a_1, \ldots, a_k\}$ be a reduced residue system modulo p^r. Prove that $a_1 \times \ldots \times a_k \equiv -1 \pmod{p^r}$.

10. (NZM 2.1 – *48). If $r_1, \ldots, r_p$ and $s_1, \ldots, s_p$ are any two complete residue systems modulo a prime $p > 2$, prove that the set $r_1s_1, \ldots, r_ps_p$ cannot be a complete residue system modulo p.