Mathematics 440/508 – Final Examination, April 17, 1997

Instructions

1. Each question is worth 20 marks.
2. Five questions are sufficient for a complete paper but you may attempt six questions for bonus marks if you wish.
3. The following notation is used throughout: $D = \{z : |z| < 1\}$, G is a open subset of the complex plane, and $H(G)$ is the set of functions analytic in G.

1. Let G consist of those complex numbers outside the two closed disks $|z+3| \leq 1$ and $|z-3| \leq 1$. Suppose that f is analytic in G. Define $A = \int_{|z+3|=2} f(z)\,dz$ and $B = \int_{|z-3|=2} f(z)\,dz$. Show that if γ is any closed path in G then there are integers m and n such that

$$\int_\gamma f(z)\,dz = mA + nB.$$

2. Suppose that f is a function which is continuous on the whole plane and analytic except possibly for $z \in [-1, 1]$. Show that f is an entire function, i.e. analytic everywhere.

3. Let $f(z) = \left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right)$, where $|a| < 1$ and $|b| < 1$. Show that f maps D two-to-one onto D, i.e. that for each $|w| < 1$, the equation $f(z) = w$ has exactly two solutions with $|z| < 1$.

4. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ maps D into D.
 (a) Show that $|a_0| \leq 1$. For which f does $|a_0| = 1$?
 (b) If $|a_0| < 1$, and we define $f_1(z) = \dfrac{f(z) - a_0}{z(1 - \bar{a}_0 f(z))}$, show that f_1 maps D into D.
 (c) Show that $|a_1| \leq 1 - |a_0|^2$. For which f does equality hold?

5. Let $a \in G$ and suppose that $f : G \to D$ is analytic, one-one and onto and has $f(a) = 0$. Suppose that $g : G \to D$ is analytic with $g(a) = 0$ but is not necessarily one-on or onto. Show that $|g(z)| \leq |f(z)|$ for all $z \in G$.

6. (a) Define "normal family of $H(G)$".
 (b) Show that if $F \in H(G)$ and if $\mathcal{F}$ is the set of $f \in H(G)$ for which $|f(z)| \leq |F(z)|$ for all $z \in G$, then $\mathcal{F}$ is a normal family in $H(G)$.