Mathematics 440/508 – Final Examination, April 10, 2002

Instructions

1. You may consult your textbook and course notes.
2. Each question is worth 20 marks.
3. Five questions are sufficient for a complete paper. If you do more than 5 questions, your mark will be based on the best 5 answers.
4. Complete answers to a few questions are preferable to incomplete answers to a larger number of questions.

Questions

1. Find all entire functions f for which $f(1/n) = 1/n^2$ for all $n = 1, 2, 3, \ldots$. Be sure to justify your answer.

2. Suppose that a function $f(z)$ is defined in $|z| < 1$ by a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence 1. Is it possible that $f(z)$ can be continued analytically to all of $\mathbb{C}$ as an entire function? Give your reasoning.

3. Show that for any integer $n > 1$, the polynomial $z^n - 3z^{n-1} - 1$ has exactly $n-1$ zeros in $|z| < 1$ and one zero in $|z| > 1$. Let α_n denote the zero in $|z| > 1$. Show that $\lim_{n\to\infty} \alpha_n = 3$.

4. Let $f(w)$ be defined by the double integral $\displaystyle\iint_{\bar{U}} \frac{1}{z-w}\, dxdy$ over the closed unit disk, $\bar{U} = \{z = x + iy : |z| \leq 1\}$. Show that $f(w)$ is an analytic function of w in the exterior domain $D = \{w : |w| > 1\} \cup \{\infty\}$. Evaluate the residue of f at ∞.

5. Let D denote the quarter disk $D = \{z = x + iy : |z| < 1, x > 0, y > 0\}$. Find a conformal map f of D onto itself which extends to the boundary ∂D and for which $f(0) = 1$, $f(1) = i$ and $f(i) = 0$. Be sure to justify your answer.

6. Let D be a simply connected domain. Show that for any two points z_1, z_2 in D there is an analytic function f mapping D one-one onto itself for which $f(z_1) = z_2$. Is f unique? Is this result true if D is not simply connected?

7. Let D and G be two domains in $\mathbb{C}$ and let $f_n(z)$ be a sequence of analytic functions such that $f_n(D) \subset G$. Suppose that $f_n(z)$ converges normally to $f(z)$ in D. Show that either $f(z)$ is a constant or else $f(D) \subset G$. (Recall that we found that the "proof" of this sketched on p.456 of the text is not correct. Try to give a correct proof.)