Mathematics 440/508 – Final Examination, April 10, 2002

Instructions
1. You may consult your textbook and course notes.
2. Each question is worth 20 marks.
3. Five questions are sufficient for a complete paper. If you do more than 5 questions, your mark will be based on the best 5 answers.
4. Complete answers to a few questions are preferable to incomplete answers to a larger number of questions.

Questions
1. Find all entire functions \( f \) for which \( f(1/n) = 1/n^2 \) for all \( n = 1, 2, 3, \ldots \). Be sure to justify your answer.

2. Suppose that a function \( f(z) \) is defined in \( |z| < 1 \) by a power series \( \sum_{n=0}^{\infty} a_n z^n \) with radius of convergence 1. Is it possible that \( f(z) \) can be continued analytically to all of \( \mathbb{C} \) as an entire function? Give your reasoning.

3. Show that for any integer \( n > 1 \), the polynomial \( z^n - 3z^{n-1} - 1 \) has exactly \( n - 1 \) zeros in \( |z| < 1 \) and one zero in \( |z| > 1 \). Let \( \alpha_n \) denote the zero in \( |z| > 1 \). Show that \( \lim_{n \to \infty} \alpha_n = 3 \).

4. Let \( f(w) \) be defined by the double integral \( \iint_{\bar{U}} \frac{1}{z-w} \, dx \, dy \) over the closed unit disk, \( \bar{U} = \{ z = x + iy : |z| \leq 1 \} \). Show that \( f(w) \) is an analytic function of \( w \) in the exterior domain \( D = \{ w : |w| > 1 \} \cup \{ \infty \} \). Evaluate the residue of \( f \) at \( \infty \).

5. Let \( D \) denote the quarter disk \( D = \{ z = x + iy : |z| < 1, x > 0, y > 0 \} \). Find a conformal map \( f \) of \( D \) onto itself which extends to the boundary \( \partial D \) and for which \( f(0) = 1, f(1) = i \) and \( f(i) = 0 \). Be sure to justify your answer.

6. Let \( D \) be a simply connected domain. Show that for any two points \( z_1, z_2 \) in \( D \) there is an analytic function \( f \) mapping \( D \) one-one onto itself for which \( f(z_1) = z_2 \). Is \( f \) unique? Is this result true if \( D \) is not simply connected?

7. Let \( D \) and \( G \) be two domains in \( \mathbb{C} \) and let \( f_n(z) \) be a sequence of analytic functions such that \( f_n(D) \subset G \). Suppose that \( f_n(z) \) converges normally to \( f(z) \) in \( D \). Show that either \( f(z) \) is a constant or else \( f(D) \subset G \). (Recall that we found that the “proof” of this sketched on p.456 of the text is not correct. Try to give a correct proof.)