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A symmetry-based method for constructing nonlocally related partial differential equation systems

George W. Bluman[a] and Zhengzheng Yang[b]
Department of Mathematics, University of British Columbia, Vancouver, British Columbia V6T 1Z2, Canada

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Nonlocally related partial differential equation (PDE) systems are important in the analysis of a given PDE system. In particular, they are useful for seeking nonlocal symmetries. It is known that each local conservation law of a given PDE system systematically yields a nonlocally related PDE system. In this paper, a new and complementary method for constructing nonlocally related PDE systems is introduced. In particular, it is shown that each point symmetry of a PDE system systematically yields a nonlocally related PDE system. Examples include nonlinear reaction-diffusion equations, nonlinear diffusion equations, and nonlinear wave equations. The considered nonlinear reaction-diffusion equations have no local conservation laws. Previously unknown nonlocal symmetries are exhibited through our new symmetry-based method for two examples of nonlinear wave equations. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4819724]

I. INTRODUCTION

A symmetry of a partial differential equation (PDE) system is any transformation of its solution manifold into itself, i.e., a symmetry transforms any solution of the PDE system to a solution of the same system. In particular, continuous symmetries of a PDE system are continuous deformations of its solutions to solutions of the same PDE system. Consequently, continuous symmetries of PDE systems are defined topologically and hence not restricted to just point or local symmetries. Thus, in principle, any nontrivial PDE system has symmetries. The problem is to find and use symmetries. Practically, to find a symmetry of a PDE system systematically, one is essentially restricted to transformations acting locally on some finite-dimensional space, whose variables are not restricted to just the independent and dependent variables of the PDE system. From this point of view, local symmetries, whose infinitesimals depend at most on a finite number of derivatives of the dependent variables of the given PDE system, constitute only a subset of the total set of symmetries of a PDE system. Otherwise, there exist nonlocal symmetries of a PDE system.[1–4] However, when one directly applies Lie's algorithm to find nonlocal symmetries, the coefficients of the infinitesimal generators should essentially involve integrals of the dependent variables and their derivatives. It is difficult to set up and obtain solutions of corresponding determining equations for such coefficients.

In Ref. 5, a systematic procedure was introduced to seek nonlocal symmetries (potential symmetries) for a given PDE system through potential systems that naturally arise from its conservation laws. A related heuristic approach to find nonlocal symmetries, called quasilocal symmetries, was presented in Refs. 6 and 7, where rich sets of examples were exhibited, especially those involving the gas dynamics equations.

An equivalent nonlocally related PDE system can play an important role in the analysis of a given PDE system. Each solution of such a nonlocally related PDE system yields a solution of the given PDE system, and, conversely, each solution of the given PDE system yields a solution of the nonlocally

[a]Electronic mail: bluman@math.ubc.ca
[b]Electronic mail: yangzz@math.ubc.ca

related PDE system. These corresponding solutions are obtained through connection formulas. More importantly, the relationship between the solutions is not one-to-one. Hence, for a given PDE system, one could be more successful when applying a standard method of analysis, especially a coordinate independent method, to a nonlocally related PDE system. For instance, through a nonlocally related PDE system, one can systematically find nonlocal symmetries and nonlocal conservation laws of a given PDE system. It turns out that such nonlocal symmetries and nonlocal conservation laws can arise as local symmetries and local conservation laws of nonlocally related PDE systems. Thus, any method depending on local symmetry analysis is valid for nonlocally related PDE systems. When nonlocal symmetries can be found for a given PDE system, it may be possible to use such symmetries systematically to generate further exact solutions from its known solutions, to construct invariant solutions, to find linearizations, or find additional nonlocally related PDE systems (see, e.g., Refs. 8–11).

A systematic procedure for finding nonlocally related PDE systems is presented in Ref. 4 and references therein. Here one constructs a tree of nonlocally related systems that consists of potential systems and subsystems. The potential systems arise naturally from local conservation laws. However, open problems remain: How can one further systematically extend a tree of nonlocally related PDE systems for a given PDE system, and, of particular importance, if the given system has no local conservation law and no known nonlocally related subsystem?

In this paper, we present a new systematic method for constructing nonlocally related PDE systems for a given PDE system through one-parameter Lie groups of point transformations that leave its solution manifold invariant (*point symmetries*). In particular, we show that a nonlocally related PDE system (*inverse potential system*) arises naturally from each point symmetry of a PDE system. As a consequence, one is able to further extend the conservation law-based method for the construction of trees of nonlocally related PDE systems. For a given PDE system, we show that nonlocally related PDE systems arising from its point symmetries can also yield nonlocal symmetries.

This paper is organized as follows. In Sec. II, we introduce the new systematic method to construct nonlocally related PDE systems. In Sec. III, the new method is used to construct nonlocally related PDE systems for nonlinear reaction-diffusion equations, nonlinear diffusion equations, and nonlinear wave equations. In Sec. IV, the point symmetries for the inverse potential systems constructed in Sec. III are shown to yield nonlocal symmetries for the considered example equations. Finally, in Sec. V, the new results in this paper are summarized and open problems are posed.

In this work, we use the package GeM for Maple[12] for symmetry and conservation law analysis.

II. NEW METHOD: NONLOCALLY RELATED PDE SYSTEMS ARISING FROM POINT SYMMETRIES

Consider a PDE system of order l with two independent variables (x, t) and m dependent variables $u = (u^1, \ldots, u^m)$ given by

$$R^\sigma[u] = R^\sigma(x, t, u, \partial u, \partial^2 u, \ldots, \partial^l u) = 0, \quad \sigma = 1, \ldots, s, \tag{2.1}$$

where $\partial^i u$ denotes the ith order partial derivative of u.

A systematic conservation law-based method for constructing nonlocally related PDE systems of the PDE system (2.1) was presented in Ref. 5. Here, the starting point is to use a nontrivial local conservation law of the PDE system (2.1):

$$D_t\Phi[u] + D_x\Psi[u] = 0. \tag{2.2}$$

Based on the conservation law (2.2), one constructs a corresponding nonlocally related PDE system (potential system) of the PDE system (2.1) given by

$$\begin{aligned} v_x &= \Phi[u],\\ v_t &= -\Psi[u],\\ R^\sigma[u] &= 0, \quad \sigma = 1, \ldots, s. \end{aligned} \tag{2.3}$$

In this paper, instead of using a conservation law as the starting point, we present a new systematic method to construct a nonlocally related PDE system through the use of an admitted point symmetry as the starting point.

Suppose the PDE system (2.1) has a point symmetry with infinitesimal generator $\mathbf{X} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \sum_{i=1}^{m}\eta^i(x,t,u)\frac{\partial}{\partial u^i}$. By introducing canonical coordinates corresponding to $\mathbf{X}$,

$$\begin{aligned} X &= X(x,t,u),\\ T &= T(x,t,u),\\ U^i &= U^i(x,t,u), \quad i=1,\dots,m, \end{aligned} \tag{2.4}$$

satisfying

$$\begin{aligned} \mathbf{X}X &= 0,\\ \mathbf{X}T &= 0,\\ \mathbf{X}U^1 &= 1,\\ \mathbf{X}U^i &= 0, \quad i=2,\dots,m, \end{aligned} \tag{2.5}$$

one maps $\mathbf{X}$ into the canonical form $\mathbf{Y} = \frac{\partial}{\partial U^1}$, while the PDE system (2.1) is mapped to an invertibly equivalent PDE system in terms of the canonical coordinates (X, T, U) with $U = (U^1, \dots, U^m)$. Since an invertible transformation maps a symmetry of a PDE system to a symmetry of the transformed system, $\mathbf{Y}$ is the infinitesimal generator of a point symmetry of the invertibly equivalent PDE system. Consequently, the invertibly equivalent PDE system is invariant under translations in U^1. It follows that the invertibly equivalent PDE system is of the form

$$\hat{R}^\sigma(X,T,\hat{U},\partial U,\dots,\partial^l U) = 0, \quad \sigma = 1,\dots,s, \tag{2.6}$$

where $\hat{U} = (U^2,\dots,U^m)$.

Introducing two new variables α and β for the first partial derivatives of U^1, one obtains the equivalent *intermediate system*

$$\begin{aligned} \alpha &= U^1_T,\\ \beta &= U^1_X,\\ \tilde{R}^\sigma&(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\dots,\partial^{l-1}\alpha,\partial^{l-1}\beta,\partial^l\hat{U}) = 0, \quad \sigma=1,\dots,s, \end{aligned} \tag{2.7}$$

where $\tilde{R}^\sigma(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\dots,\partial^{l-1}\alpha,\partial^{l-1}\beta,\partial^l\hat{U}) = 0$ is obtained from $\hat{R}^\sigma(X, T, \hat{U}, \partial U, \dots, \partial^l U) = 0$ after making the appropriate substitutions. By construction, the intermediate system (2.7) is locally related to the PDE system (2.6), and hence locally related to the given PDE system (2.1).

Excluding the dependent variable U^1 from the intermediate system (2.7), one obtains the *inverse potential system*

$$\begin{aligned} \alpha_X &= \beta_T,\\ \tilde{R}^\sigma&(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\dots,\partial^{l-1}\alpha,\partial^{l-1}\beta,\partial^l\hat{U}) = 0, \quad \sigma=1,\dots,s. \end{aligned} \tag{2.8}$$

Since the inverse potential system (2.8) is obtained by excluding U^1 from the intermediate system (2.7), and U^1 cannot be expressed as a local function of X, T and the remaining dependent variables $(\hat{U}, \alpha, \beta)$, and their derivatives, it follows that the inverse potential system (2.8) is nonlocally related to the PDE system (2.7). In particular, the intermediate system (2.7) is a potential system of the inverse potential system (2.8). Here, if $(\alpha, \beta, U^2, \dots, U^m) = (f(x,t), g(x,t), h^2(x,t), \dots, h^m(x,t))$ solves the inverse potential system (2.8), there exists a family of functions $U^1 = h^1(x,t) + C$, where C is an arbitrary constant, such that $(\alpha, \beta, U^1, \dots, U^m) = (f(x,t), g(x,t), h^1(x,t) + C, h^2(x,t), \dots, h^m(x,t))$ solves the intermediate system (2.7). By projection, $(U^1, \dots, U^m) = (h^1(x,t) + C,$

$h^2(x, t), \ldots, h^m(x, t))$ is a solution of the PDE system (2.6). Thus, the correspondence between the solutions of the inverse potential system (2.8) and those of the PDE system (2.6) is not one-to-one. It follows that the inverse potential system (2.8) is nonlocally related to the PDE system (2.6), and hence nonlocally related to the given PDE system (2.1).

Based on the above discussion, we have proved the following theorem.

Theorem 1. Any point symmetry of a PDE system (2.1) yields a nonlocally related inverse potential system given by the PDE system (2.8).

Corollary 1. Consider a scalar PDE given by

$$u_t = F(x, t, u_1, \ldots, u_n), \tag{2.9}$$

where $u_i \equiv \frac{\partial^i u}{\partial x^i}$. Let $\beta = u_x$. Then the scalar PDE,

$$\beta_t = D_x F(x, t, \beta, \ldots, \beta_{n-1}), \tag{2.10}$$

is locally related to the inverse potential system obtained from the invariance of the scalar PDE (2.9) under translations in u.

Proof. Introducing new variables α and β for the first partial derivatives of u, one obtains the intermediate system

$$\begin{aligned} \alpha &= u_t, \\ \beta &= u_x, \\ \alpha &= F(x, t, \beta, \ldots \beta_{n-1}), \end{aligned} \tag{2.11}$$

locally related to the PDE (2.9). Excluding the dependent variable u from the intermediate system (2.11), one obtains the inverse potential system

$$\begin{aligned} \alpha_x &= \beta_t, \\ \alpha &= F(x, t, \beta, \ldots, \beta_{n-1}). \end{aligned} \tag{2.12}$$

From Theorem 1, the inverse potential system (2.12) is nonlocally related to the PDE (2.9). Furthermore, one can exclude the dependent variable α from the inverse potential system (2.12) to obtain the subsystem given by the scalar PDE (2.10).

Since the excluded variable α can be expressed from the equations of the inverse potential system (2.12) in terms of β and its derivatives, the scalar PDE (2.10) is locally related to the inverse potential system (2.12). □

Remark 1. A similar relationship between the scalar PDEs (2.9) and (2.10) appears in Ref. 13.

Remark 2. Connection between the symmetry-based method and the conservation law-based method. The symmetry-based method to obtain a nonlocally related PDE system does not require the existence of a nontrivial local conservation law of a given PDE system. Thus, the new method is complementary to the conservation law-based method for constructing nonlocally related PDE systems. In particular, for the conservation law-based method, the constructed nonlocally related PDE system is a potential system of the given PDE system. For the symmetry-based method, the directly constructed intermediate system is locally related to the given PDE system. In turn, the intermediate system is a potential system of the inverse potential system. The symmetry-based method involves the reverse direction of the conservation law-based method.

Remark 3. The situation for a PDE system with at least three independent variables. The symmetry-based method can be adapted to a PDE system which has at least three independent variables. For simplicity, consider a scalar PDE with $n \geq 3$ independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable u:

$$R(x, u, \partial u, \partial^2 u, \ldots, \partial^l u) = 0. \tag{2.13}$$

The general case can be considered in a similar way. Suppose the scalar PDE (2.13) has a point symmetry with the infinitesimal generator $\mathbf{X}$. In terms of canonical coordinates given by

$$\begin{aligned} X^i &= X^i(x, u), \quad i = 1, \dots, n, \\ U &= U(x, t, u), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} \mathbf{X}X^i &= 0, \quad i = 1, \dots, n, \\ \mathbf{X}U &= 1, \end{aligned} \tag{2.15}$$

the infinitesimal generator $\mathbf{X}$ maps into the canonical form $\mathbf{Y} = \frac{\partial}{\partial U}$. In terms of (X, U) coordinates with $X = (X^1, \dots, X^n)$, the scalar PDE (2.13) becomes an invertibly related PDE of the form

$$\hat{R}(X, \partial U, \partial^2 U, \dots, \partial^l U) = 0. \tag{2.16}$$

Introducing the new variables $\alpha = (\alpha^1, \dots, \alpha^n)$ for the first partial derivatives of U, one obtains the equivalent locally related intermediate system

$$\begin{aligned} \alpha^i &= U_{X^i}, \quad i = 1, \dots, n, \\ \tilde{R}&(X, \alpha, \partial\alpha \dots, \partial^{l-1}\alpha) = 0, \end{aligned} \tag{2.17}$$

where $\tilde{R}(X, \alpha, \partial\alpha, \dots, \partial^{l-1}\alpha) = 0$ is obtained from $\hat{R}(X, \partial U, \partial^2 U, \dots, \partial^l U) = 0$ after making the appropriate substitutions. Excluding U from the intermediate system (2.17), one obtains the inverse potential system

$$\begin{aligned} &\alpha^i_{X^j} - \alpha^j_{X^i} = 0, i, \quad j = 1, \dots, n, \\ &\tilde{R}(X, \alpha, \partial\alpha, \dots, \partial^{l-1}\alpha) = 0. \end{aligned} \tag{2.18}$$

The inverse potential system (2.18) is nonlocally related to the scalar PDE (2.16), and hence nonlocally related to the scalar PDE (2.13). Moreover, since the PDE system (2.18) has curl-type conservation laws, it could possibly yield nonlocal symmetries of the scalar PDE (2.13) from local symmetries of the inverse potential system (2.18).[4,14,15]

III. EXAMPLES OF INVERSE POTENTIAL SYSTEMS

In Sec. II, we introduced a new systematic symmetry-based method for constructing nonlocally related PDE systems (inverse potential systems) of a given PDE system. In this section, we illustrate this method by several examples.

A. Nonlinear reaction-diffusion equations

Consider the class of nonlinear reaction-diffusion equations

$$u_t - u_{xx} = Q(u), \tag{3.1}$$

where the reaction term $Q(u)$ is an arbitrary constitutive function with $Q_{uu} \neq 0$. One can show that a nonlinear reaction-diffusion equation (3.1) has no nontrivial local conservation laws for any such $Q(u)$. *Thus, it is impossible to construct nonlocally related PDE systems for a nonlinear reaction-diffusion equation (3.1) by the conservation law-based method.*

On the other hand, a nonlinear reaction-diffusion equation (3.1) has point symmetries. Thus, one can construct nonlocally related PDE systems for a nonlinear reaction-diffusion equation (3.1) through the symmetry-based method introduced in Sec. II. The point symmetry classification of the class of nonlinear reaction-diffusion equations (3.1) is presented in Table I,[16] modulo its group of

TABLE I. Point symmetry classification for the class of nonlinear reaction-diffusion equations (3.1).

$Q(u)$	No.	Admitted point symmetries
Arbitrary	2	$\mathbf{X}_1=\frac{\partial}{\partial x}, \mathbf{X}_2=\frac{\partial}{\partial t}$
$u^a (a\neq 0,1)$	3	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3=u\frac{\partial}{\partial u}-(a-1)t\frac{\partial}{\partial t}-\frac{a-1}{2}x\frac{\partial}{\partial x}$
e^u	3	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_4=\frac{\partial}{\partial u}-t\frac{\partial}{\partial t}-\frac{1}{2}x\frac{\partial}{\partial x}$
$u\ln u$	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_5=ue^t\frac{\partial}{\partial u}, \mathbf{X}_6=2e^t\frac{\partial}{\partial x}-xue^t\frac{\partial}{\partial u}$

equivalence transformations

$$\begin{aligned}\bar{x}&=a_1x+a_2,\\ \bar{t}&=a_1^2t+a_3,\\ \bar{u}&=a_4u+a_5,\\ \bar{Q}&=\frac{a_4}{a_1^2}Q.\end{aligned}\tag{3.2}$$

1. The case when Q(u) is arbitrary

For arbitrary $Q(u)$, a nonlinear reaction-diffusion equation (3.1) has the two exhibited point symmetries $\mathbf{X}_1$ and $\mathbf{X}_2$. Therefore, using the symmetry-based method one can use interchanges of x and u and also t and u to construct two inverse potential systems for a nonlinear reaction-diffusion equation (3.1).

a. Inverse potential system arising from $\mathbf{X}_1$. After an interchange of the variables x and u, a nonlinear reaction-diffusion equation (3.1) becomes the invertibly related PDE given by

$$x_t=\frac{x_{uu}-Q(u)x_u^3}{x_u^2}.\tag{3.3}$$

Corresponding to the invariance of PDE (3.3) under translations of its dependent variable x, one introduces the variables v and w for the first partial derivatives of x to obtain the locally related intermediate system

$$\begin{aligned}v&=x_u,\\ w&=x_t,\\ w&=\frac{v_u-Q(u)v^3}{v^2}.\end{aligned}\tag{3.4}$$

Excluding x from the intermediate system (3.4), one obtains the inverse potential system

$$\begin{aligned}v_t&=w_u,\\ w&=\frac{v_u-Q(u)v^3}{v^2}.\end{aligned}\tag{3.5}$$

Moreover, one can exclude w from the inverse potential system (3.5) to obtain its locally related subsystem

$$v_t=\left(\frac{v_u-Q(u)v^3}{v^2}\right)_u.\tag{3.6}$$

Since the scalar PDE (3.6) is in a conservation law form and a nonlinear reaction-diffusion equation (3.1) has no local conservation laws, it follows that there is no invertible transformation that relates the scalar PDE (3.6) and the nonlinear reaction-diffusion equation (3.1). Consequently, the scalar PDE (3.6) is nonlocally related to the nonlinear reaction-diffusion equation (3.1).

b. Inverse potential system arising from $\mathbf{X}_2$. After an interchange of the variables t and u, a nonlinear reaction-diffusion equation (3.1) becomes

$$t_u^2 - Q(u)t_u^3 + t_u^2 t_{xx} - 2t_x t_u t_{xu} + t_x^2 t_{uu} = 0, \tag{3.7}$$

which is not in solved form and has mixed derivatives.

Corresponding to the invariance of PDE (3.7) under translations of its dependent variable t, one introduces two new variables $\alpha = t_x$ and $\beta = t_u$ to obtain the locally related intermediate system

$$\begin{aligned} &\alpha = t_x, \\ &\beta = t_u, \\ &\beta^2 - Q(u)\beta^3 + \beta^2\alpha_x - 2\alpha\beta\alpha_u + \alpha^2\beta_u = 0. \end{aligned} \tag{3.8}$$

Excluding t from the intermediate system (3.8), one obtains a second inverse potential system for a nonlinear reaction-diffusion equation (3.1) given by

$$\begin{aligned} &\alpha_u - \beta_x = 0, \\ &\beta^2 - Q(u)\beta^3 + \beta^2\alpha_x - 2\alpha\beta\alpha_u + \alpha^2\beta_u = 0. \end{aligned} \tag{3.9}$$

The constructed inverse potential systems for a nonlinear reaction-diffusion equation (3.1) ($Q(u)$ is arbitrary) are illustrated in Figure 1.

2. Inverse potential system arising from $\mathbf{X}_3$ when $Q(u) = u^3$

When $Q(u) = u^a$, ($a \neq 0, 1$), the corresponding class of nonlinear reaction-diffusion equations (3.1) has one additional point symmetry $\mathbf{X}_3$. For simplicity, we consider the case when $a = 3$, i.e., $Q(u) = u^3$. The general case is considered in a similar way. Canonical coordinates induced by $\mathbf{X}_3$ are given by

$$\begin{aligned} X &= xu, \\ T &= \frac{t}{x^2}, \\ U &= -\ln x. \end{aligned} \tag{3.10}$$

In (X, T, U) coordinates, the corresponding nonlinear reaction-diffusion equation (3.1) becomes the invertibly related PDE

$$\begin{aligned} &-3U_X^2 - 2XU_X^3 - X^3U_X^3 - U_X^2U_T + 10TU_X^2U_T + U_{XX} - 4TU_TU_{XX} \\ &+ 4T^2U_T^2U_{XX} + 4T^2U_X^2U_{TT} + 4TU_XU_{TX} - 8T^2U_XU_TU_{TX} = 0. \end{aligned} \tag{3.11}$$

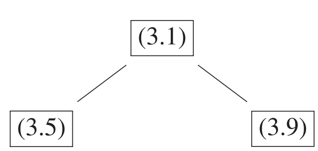

FIG. 1. Constructed inverse potential systems for a nonlinear reaction-diffusion equation (3.1) ($Q(u)$ is arbitrary).

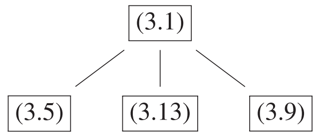

FIG. 2. Constructed inverse potential systems for the nonlinear reaction-diffusion equation (3.1) ($Q(u) = u^3$).

Accordingly, introducing the new variables $\alpha = U_X$ and $\beta = U_T$, one obtains the locally related intermediate system

$$
\begin{aligned}
&\alpha = U_X,\\
&\beta = U_T,\\
&-3\alpha^2 - 2X\alpha^3 - X^3\alpha^3 - \alpha^2\beta + 10T\alpha^2\beta + \alpha_X - 4T\beta\alpha_X\\
&+4T^2\beta^2\alpha_X + 4T^2\alpha^2\beta_T + 4T\alpha\beta_X - 8T^2\alpha\beta\beta_X = 0.
\end{aligned}
\tag{3.12}
$$

Excluding U from the intermediate system (3.12), one obtains a third inverse potential system of the corresponding nonlinear reaction-diffusion equation (3.1) given by

$$
\begin{aligned}
&\alpha_T = \beta_X,\\
&-3\alpha^2 - 2X\alpha^3 - X^3\alpha^3 - \alpha^2\beta + 10T\alpha^2\beta + \alpha_X - 4T\beta\alpha_X\\
&+4T^2\beta^2\alpha_X + 4T^2\alpha^2\beta_T + 4T\alpha\beta_X - 8T^2\alpha\beta\beta_X = 0.
\end{aligned}
\tag{3.13}
$$

The constructed inverse potential systems for the nonlinear reaction-diffusion equation (3.1) ($Q(u) = u^3$) are illustrated in Figure 2.

Moreover, since the PDE systems (3.5) and (3.13) do not have the same number of point symmetries, it follows that there is no invertible transformation relating these two PDE systems. Hence, the PDE systems (3.5) and (3.13) are nonlocally related. Similarly, the PDE systems (3.9) and (3.13) are also nonlocally related.

3. Inverse potential system arising from $\mathbf{X}_4$ *when* $Q(u) = e^u$

When $Q(u) = e^u$, the nonlinear reaction-diffusion equation (3.1) has one additional point symmetry $\mathbf{X}_4$. Canonical coordinates induced by $\mathbf{X}_4$ are given by

$$
\begin{aligned}
X &= u + 2\ln x,\\
T &= \frac{t}{x^2},\\
U &= -2\ln x.
\end{aligned}
\tag{3.14}
$$

In (X, T, U) coordinates, the corresponding nonlinear reaction-diffusion equation (3.1) becomes the invertibly related PDE

$$
\begin{aligned}
&-2U_X^2 - 2U_X^3 - e^X U_X^3 - U_X^2 U_T + 6T U_X^2 U_T + 4U_{XX} - 8T U_T U_{XX}\\
&+4T^2 U_T^2 U_{XX} + 4T^2 U_X^2 U_{TT} + 8T U_X U_{TX} - 8T^2 U_X U_T U_{TX} = 0.
\end{aligned}
\tag{3.15}
$$

It follows that the introduction of the new variables $\phi = U_X$ and $\psi = U_T$ yields the locally related intermediate system

$$
\begin{aligned}
&\phi = U_X,\\
&\psi = U_T,\\
&-2\phi^2 - 2\phi^3 - e^X\phi^3 - \phi^2\psi + 6T\phi^2\psi + 4\phi_X - 8T\psi\phi_X\\
&+4T^2\psi^2\phi_X + 4T^2\phi^2\psi_T + 8T\phi\psi_X - 8T^2\phi\psi\psi_X = 0.
\end{aligned}
\tag{3.16}
$$

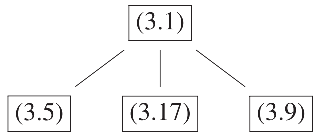

FIG. 3. Constructed inverse potential systems for the nonlinear reaction-diffusion equation (3.1) ($Q(u) = e^u$).

Excluding U from the intermediate system (3.16), one obtains a third inverse potential system of the corresponding nonlinear reaction-diffusion (3.1) given by

$$\begin{aligned} &\phi_T = \psi_X, \\ &-2\phi^2 - 2\phi^3 - e^X\phi^3 - \phi^2\psi + 6T\phi^2\psi + 4\phi_X - 8T\psi\phi_X \\ &+4T^2\psi^2\phi_X + 4T^2\phi^2\psi_T + 8T\phi\psi_X - 8T^2\phi\psi\psi_X = 0. \end{aligned} \tag{3.17}$$

The constructed inverse potential systems for the nonlinear reaction-diffusion equation (3.1) ($Q(u) = e^u$) are illustrated in Figure 3.

Moreover, since the PDE systems (3.5) and (3.17) do not have the same number of point symmetries, it follows that there is no invertible transformation relating these two PDE systems. Hence, the PDE systems (3.5) and (3.17) are nonlocally related. Similarly, the PDE systems (3.9) and (3.17) are also nonlocally related.

4. The case when $Q(u) = u \ln u$

When $Q(u) = u \ln u$, the nonlinear reaction-diffusion equation (3.1) has two additional point symmetries $\mathbf{X}_5$ and $\mathbf{X}_6$.

a. Inverse potential system arising from $\mathbf{X}_5$. Canonical coordinates induced by $\mathbf{X}_5$ are given by

$$\begin{aligned} X &= x, \\ T &= t, \\ U &= e^{-t}\ln u. \end{aligned} \tag{3.18}$$

In (X, T, U) coordinates, the corresponding nonlinear reaction-diffusion equation (3.1) becomes

$$U_T = U_{XX} + e^T U_X^2. \tag{3.19}$$

Introducing the new variables $p = U_X$ and $q = U_T$, one obtains the locally related intermediate system

$$\begin{aligned} p &= U_X, \\ q &= U_T, \\ q &= p_X + e^T p^2. \end{aligned} \tag{3.20}$$

Excluding U from the intermediate system (3.20), one obtains the inverse potential system of the corresponding nonlinear reaction-diffusion (3.1) given by

$$\begin{aligned} p_T &= q_X, \\ q &= p_X + e^T p^2. \end{aligned} \tag{3.21}$$

Moreover, excluding q from the inverse potential system (3.21), one obtains the locally related subsystem of the inverse potential system (3.21) given by

$$p_T = p_{XX} + 2e^T p p_X, \tag{3.22}$$

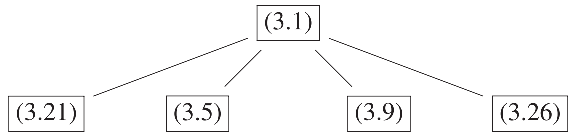

FIG. 4. Constructed inverse potential systems for the nonlinear reaction-diffusion equation (3.1) ($Q(u) = u \ln u$).

which is in a conservation law form. Since the PDE (3.22) is in a conservation law form and any nonlinear reaction-diffusion equation (3.1) has no local conservation laws, it follows that the PDE (3.22) is nonlocally related to the corresponding nonlinear reaction-diffusion equation (3.1).

b. Inverse potential system arising from $\mathbf{X}_6$. Canonical coordinates induced by $\mathbf{X}_6$ are given by

$$\begin{aligned} X &= e^{\frac{x^2}{4}} u, \\ T &= t, \\ U &= \tfrac{1}{2} e^{-t} x. \end{aligned} \tag{3.23}$$

In (X, T, U) coordinates, the corresponding nonlinear reaction-diffusion equation (3.1) becomes

$$U_T = \frac{e^{-2T} U_{XX} + 2XU_X^3 - 4X \ln X U_X^3}{4U_X^2}. \tag{3.24}$$

Introducing the new variables $r = U_X$ and $s = U_T$, one obtains the locally related intermediate system

$$\begin{aligned} r &= U_X, \\ s &= U_T, \\ s &= \frac{e^{-2T} r_X + 2Xr^3 - 4X \ln X r^3}{4r^2}. \end{aligned} \tag{3.25}$$

Excluding U from the intermediate system (3.25), one obtains the inverse potential system of the corresponding nonlinear reaction-diffusion (3.1) given by

$$\begin{aligned} r_T &= s_X, \\ s &= \frac{e^{-2T} r_X + 2Xr^3 - 4X \ln X r^3}{4r^2}. \end{aligned} \tag{3.26}$$

Excluding s from the inverse potential system (3.26), one obtains the locally related subsystem of the inverse potential system (3.26) given by

$$r_T = \left(\frac{e^{-2T} r_X + 2Xr^3 - 4X \ln X r^3}{4r^2} \right)_X, \tag{3.27}$$

which is in a conservation law form.

The constructed inverse potential systems for the nonlinear reaction-diffusion equation (3.1) ($Q(u) = u \ln u$) are illustrated in Figure 4.

B. Nonlinear diffusion equations

As a second example, consider the class of scalar nonlinear diffusion equations

$$v_t = K(v_x) v_{xx}, \tag{3.28}$$

where $K(v_x)$ is an arbitrary nonconstant constitutive function. The point symmetry classification of its locally related class of PDE systems,

$$\begin{aligned} v_x &= u, \\ v_t &= K(u) u_x, \end{aligned} \tag{3.29}$$

TABLE II. Point symmetry classification for the class of PDE systems (3.29).

$K(u)$	No.	Admitted point symmetries
Arbitrary	4	$\mathbf{Y}_1 = \frac{\partial}{\partial x}$, $\mathbf{Y}_2 = \frac{\partial}{\partial t}$, $\mathbf{Y}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$, $\mathbf{Y}_4 = \frac{\partial}{\partial v}$
$u^\mu (\mu \neq 0)$	5	$\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_5 = x\frac{\partial}{\partial x} + \frac{2}{\mu}u\frac{\partial}{\partial u} + \left(1+\frac{2}{\mu}\right)v\frac{\partial}{\partial v}$
e^u	5	$\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_6 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u} + (2x+v)\frac{\partial}{\partial v}$
u^{-2}	∞	$\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_5\ (\mu = -2)$, $\mathbf{Y}_7 = -xv\frac{\partial}{\partial x} + (xu+v)u\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial v}$, $\mathbf{Y}_8 = -x(2t+v^2)\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} + u(6t+2xuv+v^2)\frac{\partial}{\partial u} + 4tv\frac{\partial}{\partial v}$, $\mathbf{Y}_\infty = F(v,t)\frac{\partial}{\partial x} - u^2G(v,t)\frac{\partial}{\partial u}$, where $(F(v,t), G(v,t))$ is an arbitrary solution of the linear system: $F_t = G_v$, $F_v = G$
$\frac{1}{1+u^2}e^{\lambda \arctan u}$	5	$\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4$, $\mathbf{Y}_9 = v\frac{\partial}{\partial x} + \lambda t\frac{\partial}{\partial t} - \left(1+u^2\right)\frac{\partial}{\partial u} - x\frac{\partial}{\partial v}$

is listed in Table II,[4,17] modulo its group of equivalence transformations given by

$$\begin{aligned} \bar{t} &= a_1 t + a_2, \\ \bar{x} &= a_3 x + a_4 v + a_5, \\ \bar{u} &= \frac{a_6 + a_7 u}{a_3 + a_4 u}, \\ \bar{v} &= a_6 x + a_7 v + a_8, \\ \bar{K} &= \frac{(a_3 + a_4 u)^2}{a_1} K, \end{aligned} \tag{3.30}$$

where $a_1, \ldots, a_8$ are arbitrary constants with $a_1(a_3a_7 - a_4a_6) \neq 0$.

By projection of the symmetries in Table II, one sees that for arbitrary $K(v_x)$, there are four point symmetries of a nonlinear diffusion equation (3.28), namely, $\mathbf{Y}_1 = \frac{\partial}{\partial x}$, $\mathbf{Y}_2 = \frac{\partial}{\partial t}$, $\mathbf{Y}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$, and $\mathbf{Y}_4 = \frac{\partial}{\partial v}$.

1. Inverse potential system arising from $\mathbf{Y}_1$

Since a nonlinear diffusion equation (3.28) is invariant under translations of its independent variable x, one can interchange x and v to generate an invertibly related PDE of a nonlinear diffusion equation (3.28) given by

$$x_t = \frac{K\left(\frac{1}{x_v}\right) x_{vv}}{x_v^2}. \tag{3.31}$$

Introducing new variables $w = x_v$ and $y = x_t$, one obtains the locally related intermediate system

$$\begin{aligned} w &= x_v, \\ y &= x_t, \\ y &= \frac{K\left(\frac{1}{w}\right) w_v}{w^2}. \end{aligned} \tag{3.32}$$

Excluding x from the intermediate system (3.32), one obtains the inverse potential system

$$\begin{aligned} w_t &= y_v, \\ y &= \frac{K\left(\frac{1}{w}\right) w_v}{w^2}. \end{aligned} \tag{3.33}$$

Moreover, one can exclude the variable y from the inverse potential system (3.33) to obtain the locally related subsystem of the inverse potential system (3.33) given by

$$w_t = \left(\frac{K\left(\frac{1}{w}\right) w_v}{w^2} \right)_v . \tag{3.34}$$

2. Inverse potential system arising from $\mathbf{Y}_2$

Since a nonlinear diffusion equation (3.28) is invariant under translations of its independent variable t, one can interchange t and v to obtain an invertibly related PDE given by

$$t_v^2 - K\left(-\frac{t_x}{t_v}\right)\left(2t_v t_x t_{xv} - t_x^2 t_{vv} - t_v^2 t_{xx}\right) = 0. \tag{3.35}$$

Introducing new variables $\alpha = t_v$ and $\beta = t_x$, one obtains the locally related intermediate system

$$\begin{aligned} &\alpha = t_v, \\ &\beta = t_x, \\ &\alpha^2 - K\left(-\frac{\beta}{\alpha}\right)\left(2\alpha\beta\alpha_x - \beta^2\alpha_v - \alpha^2\beta_x\right) = 0. \end{aligned} \tag{3.36}$$

Excluding t from the intermediate system (3.36), one obtains the inverse potential system

$$\begin{aligned} &\alpha_x = \beta_v, \\ &\alpha^2 - K\left(-\frac{\beta}{\alpha}\right)\left(2\alpha\beta\alpha_x - \beta^2\alpha_v - \alpha^2\beta_x\right) = 0. \end{aligned} \tag{3.37}$$

3. Inverse potential system arising from $\mathbf{Y}_3$

Since a nonlinear diffusion equation (3.28) is invariant under the scaling symmetry generated by $\mathbf{Y}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$, one can use a corresponding canonical coordinate transformation given by

$$\begin{aligned} X &= \frac{t}{x^2}, \\ T &= \frac{v}{x}, \\ V &= \ln x \end{aligned} \tag{3.38}$$

to map a nonlinear diffusion equation (3.28) into the invertibly related PDE

$$\begin{aligned} &- V_X V_T^2 + K\left(\frac{1 + T V_T + 2X V_X}{V_T}\right)\left(-4X V_T V_{TX} + V_{TT} + 4X V_X V_{TT} - V_T^2\right. \\ &\left. - 8X^2 V_X V_T V_{TX} + 4X^2 V_X^2 V_{TT} + 2X V_X V_T^2 + 4X^2 V_T^2 V_{XX}\right) = 0. \end{aligned} \tag{3.39}$$

Introducing new variables $\phi = V_X$ and $\psi = V_T$, one obtains the locally related intermediate system

$$\begin{aligned} &\phi = V_X, \\ &\psi = V_T, \\ &- \phi\psi^2 + K\left(\frac{1 + T\psi + 2X\phi}{\psi}\right)\left(-4X\psi\psi_X + \psi_T + 4X\phi\psi_T - \psi^2\right. \\ &\left. -8X^2\phi\psi\psi_X + 4X^2\phi^2\psi_T + 2X\phi\psi^2 + 4X^2\psi^2\phi_X\right) = 0. \end{aligned} \tag{3.40}$$

Excluding V from the intermediate system (3.40), one obtains the inverse potential system

$$\begin{aligned}&\phi_T = \psi_X,\\ &-\phi\psi^2 + K\left(\frac{1+T\psi+2X\phi}{\psi}\right)\left(-4X\psi\psi_X + \psi_T + 4X\phi\psi_T - \psi^2\right.\\ &\left.-8X^2\phi\psi\psi_X + 4X^2\phi^2\psi_T + 2X\phi\psi^2 + 4X^2\psi^2\phi_X\right) = 0.\end{aligned} \tag{3.41}$$

4. Inverse potential system arising from $\mathbf{Y}_4$

From its invariance under translations of its dependent variable v, one can apply directly the symmetry-based method to a nonlinear diffusion equation (3.28). Letting $u = v_x, z = v_t$, one obtains the corresponding locally related intermediate system

$$\begin{aligned}u &= v_x,\\ z &= v_t,\\ z &= K(u)u_x.\end{aligned} \tag{3.42}$$

Excluding v from the intermediate system (3.42), one obtains the inverse potential system

$$\begin{aligned}u_t &= z_x,\\ z &= K(u)u_x.\end{aligned} \tag{3.43}$$

Excluding z from the inverse potential system (3.43), one obtains the locally related subsystem of the inverse potential system (3.43) given by the class of nonlinear diffusion equations

$$u_t = (K(u)\,u_x)_x\,. \tag{3.44}$$

5. Inverse potential system for a nonlinear diffusion equation (3.44)

Now take a nonlinear diffusion equation (3.44) as the given PDE. The point symmetry classification for the class of nonlinear diffusion equations (3.44) is presented in Table III,[18] modulo its group of equivalence transformations given by

$$\begin{aligned}\bar{t} &= a_4 t + a_1,\\ \bar{x} &= a_5 x + a_2,\\ \bar{u} &= a_6 u + a_3,\\ \bar{K} &= \frac{a_5^2}{a_4}K,\end{aligned} \tag{3.45}$$

where $a_1, \ldots, a_6$ are arbitrary constants with $a_4a_5a_6 \neq 0$.

There are three point symmetries of a nonlinear diffusion equation (3.44) for arbitrary $K(u)$: $\mathbf{X}_1 = \frac{\partial}{\partial x}$, $\mathbf{X}_2 = \frac{\partial}{\partial t}$, and $\mathbf{X}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$. Therefore, one can construct three inverse potential

TABLE III. Point symmetry classification for the class of nonlinear diffusion equations (3.44).

$K(u)$	No.	Admitted point symmetries
Arbitrary	3	$\mathbf{X}_1 = \frac{\partial}{\partial x}, \mathbf{X}_2 = \frac{\partial}{\partial t}, \mathbf{X}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$
$u^\mu (\mu \neq 0)$	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 = x\frac{\partial}{\partial x} + \frac{2}{\mu}u\frac{\partial}{\partial u}$
e^u	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_5 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}$
$u^{-\frac{4}{3}}$	5	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\ (\mu = -\frac{4}{3}), \mathbf{X}_6 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$

systems for a nonlinear diffusion equation (3.44) through the symmetry-based method. Take $\mathbf{X}_1$ for example. From its invariance under translations in x, one can employ the hodograph transformation interchanging x and u to obtain the invertibly related PDE

$$x_t = -\left(\frac{K(u)}{x_u}\right)_u. \tag{3.46}$$

Accordingly, letting $p = x_u$ and $q = x_t$, one obtains the locally related intermediate system

$$\begin{aligned} p &= x_u, \\ q &= x_t, \\ q &= -\left(\frac{K(u)}{p}\right)_u. \end{aligned} \tag{3.47}$$

Excluding the variable x from the intermediate system (3.47), one obtains the inverse potential system

$$\begin{aligned} p_t &= q_u, \\ q &= -\left(\frac{K(u)}{p}\right)_u. \end{aligned} \tag{3.48}$$

Finally, after excluding the variable q from the inverse potential system (3.48), one obtains the locally related subsystem of the inverse potential system (3.48) given by

$$p_t = -\left(\frac{K(u)}{p}\right)_{uu}. \tag{3.49}$$

The constructed inverse potential systems for a nonlinear diffusion equation (3.28) ($K(v_x)$ is arbitrary) are illustrated in Figure 5.

C. Nonlinear wave equations

As a third example, consider the class of nonlinear wave equations

$$u_{tt} = (c^2(u)u_x)_x, \tag{3.50}$$

with an arbitrary nonconstant constitutive function $c(u)$.

In Refs. 4 and 10, it is shown that one can apply any invertible transformation to a PDE system with two or more dependent variables to seek additional nonlocally related subsystems of the given PDE system from exclusions of the resulting dependent variables. Theorem 1 shows that the use of an invertible transformation that is a point symmetry of a given PDE system yields a nonlocally related PDE system (inverse potential system). We now use point symmetries of the potential system of a nonlinear wave equation (3.50) given by

$$\begin{aligned} v_x &= u_t, \\ v_t &= c^2(u)u_x \end{aligned} \tag{3.51}$$

to obtain additional nonlocally related PDE systems for a nonlinear wave equation (3.50). For arbitrary $c(u)$, the potential system (3.51) has the point symmetries $\mathbf{Y}_1 = \frac{\partial}{\partial t}$, $\mathbf{Y}_2 = \frac{\partial}{\partial x}$, $\mathbf{Y}_3 = \frac{\partial}{\partial v}$,

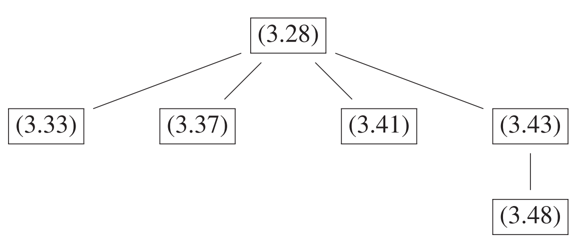

FIG. 5. Constructed inverse potential systems for a nonlinear diffusion equation (3.28) ($K(v_x)$ is arbitrary).

$\mathbf{Y}_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$, and $\mathbf{Y}_\infty$, where $\mathbf{Y}_\infty$ represents the infinite number of point symmetries arising from the linearization of the potential system (3.51) through the hodograph transformation (interchange of independent and dependent variables).

Due to its invariance under translations in v and t, for arbitrary $c(u)$, the potential system (3.51) has the point symmetry with the infinitesimal generator $\frac{\partial}{\partial v} - \frac{\partial}{\partial t}$. Corresponding canonical coordinates yield the invertible point transformation

$$\rho : \begin{cases} X = x, \\ T = u, \\ U = t + v, \\ V = v. \end{cases} \tag{3.52}$$

The point transformation (3.52) maps a potential system (3.51) into the invertibly related PDE system

$$\begin{aligned} &V_X U_T - V_T U_X - 1 = 0, \\ &V_T + c^2(T) U_X - c^2(T) V_X = 0, \end{aligned} \tag{3.53}$$

which is invariant under translations in U and V.

From the invariance of a PDE system (3.53) under translations in V, one introduces two new variables A and B for the first partial derivatives of V to obtain the intermediate system

$$\begin{aligned} &A = V_X, \\ &B = V_T, \\ &A U_T - B U_X - 1 = 0, \\ &B + c^2(T) U_X - c^2(T) A = 0. \end{aligned} \tag{3.54}$$

Excluding V from the intermediate system (3.54), one obtains the inverse potential system

$$\begin{aligned} &A_T = B_X, \\ &A U_T - B U_X - 1 = 0, \\ &B + c^2(T) U_X - c^2(T) A = 0. \end{aligned} \tag{3.55}$$

Since one can solve for A and B from the last two equations of the inverse potential system (3.55), it is straightforward to exclude A and B from the inverse potential system (3.55) to obtain its locally related scalar PDE

$$\begin{aligned} &U_{TT} + c^2(T)\left(c^2(T) U_{XX} - U_{XX} U_T^2 - U_{TT} U_X^2 - 2U_{TX} + 2U_{TX} U_T U_X\right) \\ &- 2c(T)c'(T)\left(U_X - U_X^2 U_T\right) = 0. \end{aligned} \tag{3.56}$$

In Sec. IV, we prove that the PDE (3.56) is nonlocally related to the nonlinear wave equation (3.50) through the symmetry classifications of these two classes of PDEs.

Remark 4. Another equivalent straightforward method to obtain the scalar PDE (3.56) is by excluding V directly from the PDE system (3.53) through cross-differentiation.

The coordinates in the point transformation (3.52) are a choice of canonical coordinates corresponding to the point symmetry $\mathbf{Y}_1$. The inverse potential system, arising from the invariance of the PDE system (3.53) under translations in U, yields the locally related scalar PDE

$$c(u)\left(v_x^2 v_{uu} - 2v_x v_u v_{ux} + v_{xx} v_u^2 - c^2(u) v_{xx}\right) - 2c'(u) v_x^2 v_u = 0. \tag{3.57}$$

By interchanging x and v, it is straightforward to show that the PDE (3.57) is invertibly related to the linear wave equation constructed in Ref. 10:

$$x_{vv} = \left(c^{-2}(u) x_u\right)_u. \tag{3.58}$$

IV. EXAMPLES OF NONLOCAL SYMMETRIES ARISING FROM THE SYMMETRY-BASED METHOD

In the framework of nonlocally related PDE systems, nonlocal symmetries of a given PDE system (2.1) can arise from point symmetries of any PDE system in a tree of nonlocally related PDE systems that includes (2.1).

In the conservation law-based method, from constructed nonlocally related PDE systems, three different types of nonlocal symmetries can be sought for a given PDE system (2.1):[4]

1. Nonlocal symmetries arising from point symmetries of potential systems of (2.1).
2. Nonlocal symmetries arising from point symmetries of nonlocally related subsystems of (2.1).
3. Nonlocal symmetries arising from point symmetries of nonlocally related subsystems of potential systems of (2.1).

For Type 1, a point symmetry of a potential system of (2.1) yields a nonlocal symmetry of (2.1) if and only if the infinitesimal components corresponding to its given variables (x, t, u) involve the nonlocal variables of the potential system. For Types 2 and 3, one must trace back to see whether the obtained point symmetry yields a nonlocal symmetry of (2.1).

In the symmetry-based method, one can seek further nonlocal symmetries arising from point symmetries of the constructed inverse potential systems as well as their subsystems.

In Sec. III, we constructed several inverse potential systems for nonlinear reaction-diffusion equations (3.1), nonlinear diffusion equations (3.28) and (3.44), and nonlinear wave equations (3.50). For a nonlinear reaction-diffusion equation (3.1), one can show that each point symmetry of the constructed inverse potential systems yields no nonlocal symmetry of (3.1). In this section, it is shown that for nonlinear diffusion equations (3.28) and (3.44), and nonlinear wave equations (3.50), nonlocal local symmetries do arise from some of the constructed inverse potential systems (or the locally related subsystems of such inverse potential systems). Most importantly, some previously unknown nonlocal symmetries are obtained for the nonlinear wave equation (3.50) when $c(u) = u^{-2}$ or $c(u) = u^{-\frac{2}{3}}$.

A. Nonlocal symmetries of nonlinear diffusion equations

In Tables II and III, we presented the point symmetry classifications for the classes of nonlinear diffusion equations (3.29) and (3.44).

Proposition 1. The symmetry $\mathbf{X}_6$ yields a nonlocal symmetry of the corresponding nonlinear diffusion equation (3.28) with $K(u) = u^{-\frac{4}{3}}$.

Proof. Suppose the symmetry $\mathbf{X}_6$ yields a local symmetry of the nonlinear diffusion equation (3.28) with $K(u) = u^{-\frac{4}{3}}$. Since the nonlinear diffusion equation (3.28) and the potential system (3.29) are locally related, $\mathbf{X}_6$ must also yield a local symmetry $\tilde{\mathbf{X}}_6$ of the potential system (3.29). Consequently, there must exist a differential function $f[u, v]$ such that, in evolutionary form, $\tilde{\mathbf{X}}_6 = (-3xu - x^2u_x)\frac{\partial}{\partial u} + f[u, v]\frac{\partial}{\partial v}$ is a local symmetry of the potential system (3.29). Since $v_x = u$, $v_t = u^{-\frac{4}{3}}u_x$, and $u_t = (u^{-\frac{4}{3}}u_x)_x$, one can restrict $f[u, v]$ to be of the form $f(x, t, u, v, u_x, u_{xx}, \ldots)$ depending on x, t, u and the partial derivatives of u with respect to x. First, suppose $f[u, v]$ is of the form $f(x, t, u, v, u_x)$. Applying $\tilde{\mathbf{X}}_6^{(\infty)}$ to the potential system (3.29), one obtains

$$\begin{aligned} &f_x + f_u u_x + f_v v_x + f_{u_x} u_{xx} = -3xu - x^2u_x, \\ &f_t + f_u u_t + f_v v_t + f_{u_x} u_{tx} = \tfrac{4}{3}(3xu + x^2u_x)u^{-\frac{7}{3}}u_x + D_x(-3xu - x^2u_x)u^{-\frac{4}{3}} \end{aligned} \tag{4.1}$$

on every solution of the potential system (3.29). After making appropriate substitutions and equating the coefficients of the term u_{xx}, one obtains $f_{u_x} = 0$. By similar reasoning, one can show that $f(x, t, u, v, u_x, u_{xx}, \ldots)$ has no dependence on any partial derivative of u with respect to x. Hence, $f[u, v]$ is of the form $f(x, t, u, v)$. Consequently, if $\mathbf{X}_6$ yields a local symmetry of the nonlinear diffusion equation (3.28) with $K(u) = u^{-\frac{4}{3}}$, then $\tilde{\mathbf{X}}_6$ must be a point symmetry of the corresponding potential system (3.29).

Comparing Tables II and III, one immediately sees that symmetry $\mathbf{X}_6$ does not yield a point symmetry of the corresponding potential system (3.29). This follows from the fact that when $K(u) = u^{-\frac{4}{3}}$, the potential system (3.29) has no point symmetry whose infinitesimal components corresponding to the variables (x, t) are the same as those for $\mathbf{X}_6$. Hence, $\mathbf{X}_6$ yields a nonlocal symmetry of the nonlinear diffusion equation (3.28) with $K(u) = u^{-\frac{4}{3}}$. □

Now consider the class of scalar PDEs (3.34). The equivalence transformations for this class arise from the six infinitesimal generators

$$\begin{aligned} \mathbf{E}_1 &= \frac{\partial}{\partial v}, \mathbf{E}_2 = \frac{\partial}{\partial w} + \frac{2K}{w}\frac{\partial}{\partial K}, \mathbf{E}_3 = w\frac{\partial}{\partial w} + 2K\frac{\partial}{\partial K}, \\ \mathbf{E}_4 &= v\frac{\partial}{\partial v} + 2K\frac{\partial}{\partial K}, \mathbf{E}_5 = t\frac{\partial}{\partial t} - K\frac{\partial}{\partial K}, \mathbf{E}_6 = \frac{\partial}{\partial t}. \end{aligned} \tag{4.2}$$

Thus, the group of equivalence transformations for the class of PDEs (3.34) is given by

$$\begin{aligned} \bar{v} &= a_3 v + a_1, \\ \bar{t} &= a_5 t + a_6, \\ \bar{w} &= a_4 w + a_2, \\ \bar{K} &= \frac{a_3^2 (a_4 w + a_2)^2}{a_5 w^2} K, \end{aligned} \tag{4.3}$$

where $a_1, \ldots, a_6$ are arbitrary constants with $a_3 a_4 a_5 \neq 0$.

In Table IV, we present the point symmetry classification for the class of PDEs (3.34), modulo its group of equivalence transformations (4.3).

By similar reasoning as in the proof of Proposition 1, one can show that, for $K(u) = u^{-\frac{2}{3}}$, the point symmetry $\mathbf{V}_5$ of the PDE (3.34) yields a nonlocal symmetry of the corresponding intermediate system (3.32), which is locally related to the nonlinear diffusion equation (3.28). Hence, $\mathbf{V}_5$ yields a nonlocal symmetry of the nonlinear diffusion equation (3.28) with $K(u) = u^{-\frac{2}{3}}$.

Moreover, comparing Tables III and IV, one also sees that when $K(u) = u^{-\frac{2}{3}}$, since its infinitesimal component for the variable u has an essential dependence on the variable v, the symmetry $\mathbf{V}_5$ of the corresponding PDE (3.34) yields a nonlocal symmetry of the nonlinear diffusion equation (3.44), which cannot be obtained through its potential system (3.29). By similar reasoning, when $K(u) = u^{-2}$, one can show that the symmetries $\mathbf{V}_6$, $\mathbf{V}_7$, and $\mathbf{V}_\infty$ of the PDE (3.34) yield nonlocal symmetries of the corresponding nonlinear diffusion equation (3.44). In addition, when $K(u) = e^{\frac{1}{u}} u^{-2}$,

TABLE IV. Point symmetry classification for the class of PDEs (3.34).

$K\,(1/w)$	$K(u)$	No.	Admitted point symmetries in (t, v, w) coordinates	Admitted point symmetries in (t, v, u) coordinates
Arbitrary	Arbitrary	3	$\mathbf{V}_1 = \frac{\partial}{\partial t}, \mathbf{V}_2 = \frac{\partial}{\partial v}, \mathbf{V}_3 = 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$
$w^{-\mu}$	u^{μ}	4	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3,$ $\mathbf{V}_4 = (2+\mu)v\frac{\partial}{\partial v} - 2w\frac{\partial}{\partial w}$	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3,$ $\mathbf{V}_4 = (2+\mu)v\frac{\partial}{\partial v} + 2u\frac{\partial}{\partial u}$
$w^{\frac{2}{3}}$	$u^{-\frac{2}{3}}$	5	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\ (\mu = -\frac{2}{3}),$ $\mathbf{V}_5 = 3vw\frac{\partial}{\partial w} - v^2\frac{\partial}{\partial v}$	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\ (\mu = -\frac{2}{3}),$ $\mathbf{V}_5 = -3uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}$
w^2	u^{-2}	∞	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\ (\mu = -2),$ $\mathbf{V}_6 = -vw\frac{\partial}{\partial w} + 2t\frac{\partial}{\partial v},$ $\mathbf{V}_7 = 4t^2\frac{\partial}{\partial t} + 4vt\frac{\partial}{\partial v} - (2t+v^2)w\frac{\partial}{\partial w},$ $\mathbf{V}_\infty = G(t,v)\frac{\partial}{\partial w}$, where $G(t,v)$ satisfies $G_t = G_{vv}$	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4\ (\mu = -2),$ $\mathbf{V}_6 = uv\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial v},$ $\mathbf{V}_7 = 4t^2\frac{\partial}{\partial t} + 4vt\frac{\partial}{\partial v} + (2t+v^2)u\frac{\partial}{\partial u},$ $\mathbf{V}_\infty = -u^2 G(t,v)\frac{\partial}{\partial u}$, where $G(t,v)$ satisfies $G_t = G_{vv}$
$e^w w^2$	$e^{\frac{1}{u}} u^{-2}$	4	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_8 = v\frac{\partial}{\partial v} + 2\frac{\partial}{\partial w}$	$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_8 = v\frac{\partial}{\partial v} - 2u^2\frac{\partial}{\partial u}$

one can show that $\mathbf{V}_8$ yields a point symmetry $\tilde{\mathbf{V}}_8 = (x + 2v)\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - 2u^2\frac{\partial}{\partial u}$ of the potential system (3.29) whose infinitesimal component for the variable x has an essential dependence on the variable v. Consequently, $\mathbf{V}_8$ yields a nonlocal symmetry of the nonlinear diffusion equation (3.44) when $K(u) = e^{\frac{1}{u}}u^{-2}$.

Next, consider the class of PDEs (3.49). The equivalence transformations for this class arise from the six infinitesimal generators

$$\begin{aligned} &\mathbf{E}_1 = \frac{\partial}{\partial u}, \mathbf{E}_2 = u\frac{\partial}{\partial u} + 2K\frac{\partial}{\partial K}, \mathbf{E}_3 = p\frac{\partial}{\partial p} + 2K\frac{\partial}{\partial K}, \\ &\mathbf{E}_4 = t\frac{\partial}{\partial t} - K\frac{\partial}{\partial K}, \mathbf{E}_5 = \frac{\partial}{\partial t}, \mathbf{E}_6 = u^2\frac{\partial}{\partial u} - 3up\frac{\partial}{\partial p} - 2Ku\frac{\partial}{\partial K}. \end{aligned} \tag{4.4}$$

Correspondingly, the six-parameter group of equivalence transformations for the PDE class (3.49) is given by

$$\begin{aligned} \bar{u} &= a_2u + a_1, \\ \bar{t} &= a_4t + a_5, \\ \bar{p} &= a_3p, \\ \bar{K} &= \frac{a_2^2a_3^2}{a_4}K, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \bar{u} &= \frac{u}{1 - a_6u}, \\ \bar{t} &= t, \\ \bar{p} &= (1 - a_6u)^3p, \\ \bar{K} &= (1 - a_6u)^2K, \end{aligned} \tag{4.6}$$

where $a_1, \ldots, a_6$ are arbitrary constants with $a_2a_3a_4 \neq 0$.

In Table V, we present the point symmetry classification for the class of PDEs (3.49), modulo its group of equivalence transformations given by (4.5) and (4.6).

Similar to the situation in Proposition 1, when $K(u) = \frac{1}{1+u^2}e^{\lambda \arctan u}$, the point symmetry $\mathbf{W}_5$ of the PDE (3.49) yields a nonlocal symmetry of the corresponding intermediate system (3.47), which is locally related to the nonlinear diffusion equation (3.44). Hence, $\mathbf{W}_5$ yields a nonlocal symmetry of the nonlinear diffusion equation (3.44) with $K(u) = \frac{1}{1+u^2}e^{\lambda \arctan u}$. By similar reasoning, the symmetry $\mathbf{W}_6$ also yields a nonlocal symmetry of the nonlinear diffusion equation (3.44) with $K(u) = u^{-2}$.

Taking the equivalence transformation (4.6) into consideration, one can obtain more nonlocal symmetries for the class of nonlinear diffusion equations (3.44) from the corresponding class of PDEs (3.49). In particular, the equivalence transformation (4.6) maps u^μ into $\bar{u}^\mu(1 + a_6\bar{u})^{-(\mu+2)}$, e^u into $(1 + a_6\bar{u})^{-2}e^{\frac{\bar{u}}{1+a_6\bar{u}}}$. Moreover, the symmetries $\mathbf{W}_3$ and $\mathbf{W}_4$ are mapped into $\bar{\mathbf{W}}_3$ and $\bar{\mathbf{W}}_4$, respectively.

TABLE V. Point symmetry classification for the class of PDEs (3.49).

$K(u)$	No.	Admitted point symmetries
Arbitrary	2	$\mathbf{W}_1 = \frac{\partial}{\partial t}, \mathbf{W}_2 = 2t\frac{\partial}{\partial t} + p\frac{\partial}{\partial p}$
u^μ	3	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3 = 2u\frac{\partial}{\partial u} + (\mu - 2)p\frac{\partial}{\partial p}$
e^u	3	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_4 = 2\frac{\partial}{\partial u} + p\frac{\partial}{\partial p}$
$\frac{1}{1+u^2}e^{\lambda \arctan u}$	3	$\mathbf{W}_1, \mathbf{W}_2,$ $\mathbf{W}_5 = 2(1 + u^2)\frac{\partial}{\partial u} - p(6u - \lambda)\frac{\partial}{\partial p}$
u^{-2}	4	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\ (\mu = -2), \mathbf{W}_6 = u^2\frac{\partial}{\partial u} - 3pu\frac{\partial}{\partial p}$

One can show that when $K(u) = u^{\mu}(1 + a_6u)^{-(\mu + 2)}$, $\bar{\mathbf{W}}_3 = 2u(1 + a_6u)\frac{\partial}{\partial u} - p(6a_6u - \mu + 2)\frac{\partial}{\partial p}$; when $K(u) = (1 + a_6u)^{-2}e^{\frac{u}{1+a_6u}}$, $\bar{\mathbf{W}}_4 = 2(1 + a_6u)^2\frac{\partial}{\partial u} - p(6a_6^2u + 6a_6 - 1)\frac{\partial}{\partial p}$. Similar to the situation in Proposition 1, one can show that $\bar{\mathbf{W}}_3$ and $\bar{\mathbf{W}}_4$ yield nonlocal symmetries of the corresponding nonlinear diffusion equations (3.44).

Remark 5. Comparing Tables II and V, one concludes that when $K(u) = \frac{1}{1+u^2}e^{\lambda \arctan u}$, the nonlocal symmetry yielded by $\mathbf{W}_5$ corresponds to the nonlocal symmetry yielded by $\mathbf{Y}_9$. When $K(u) = u^{-2}$, the nonlocal symmetry yielded by $\mathbf{W}_6$ corresponds to a nonlocal symmetry yielded by $\mathbf{Y}_{\infty}$.

B. Nonlocal symmetries of nonlinear wave equations

We now use the subsystem (3.56), locally related to the inverse potential system (3.55), to obtain previously unknown nonlocal symmetries for the class of nonlinear wave equations (3.50).

In Ref. 19, the point symmetry classification was obtained for the class of nonlinear wave equations (3.50), which is presented in Table VI, modulo its group of equivalence transformations

$$\begin{aligned} \bar{x} &= a_1x + a_4, \\ \bar{t} &= a_2t + a_5, \\ \bar{u} &= a_3u + a_6, \\ \bar{c} &= \frac{a_1}{a_2}c, \end{aligned} \tag{4.7}$$

where $a_1, \ldots, a_6$ are arbitrary constants with $a_1a_2a_3 \neq 0$.

The equivalence transformations for the PDE class (3.56) arise from the five infinitesimal generators

$$\begin{aligned} &\mathbf{E}_1 = \frac{\partial}{\partial T}, \mathbf{E}_2 = \frac{\partial}{\partial X}, \mathbf{E}_3 = \frac{\partial}{\partial U}, \\ &\mathbf{E}_4 = T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X} + U\frac{\partial}{\partial U}, \mathbf{E}_5 = -T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X} + c\frac{\partial}{\partial c}. \end{aligned} \tag{4.8}$$

Correspondingly, the five-parameter group of equivalence transformations for the class of PDEs (3.56) is given by

$$\begin{aligned} \bar{T} &= \frac{a_4}{a_5}T + a_1, \\ \bar{X} &= a_4a_5X + a_2, \\ \bar{U} &= a_4U + a_3, \\ \bar{c} &= a_5c, \end{aligned} \tag{4.9}$$

where $a_1, \ldots, a_5$ are arbitrary constants with $a_4a_5 \neq 0$.

TABLE VI. Point symmetry classification for the class of nonlinear wave equations (3.50).

$c(u)$	No.	Admitted point symmetries
Arbitrary	3	$\mathbf{X}_1 = \frac{\partial}{\partial x}, \mathbf{X}_2 = \frac{\partial}{\partial t}, \mathbf{X}_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$
u^{μ}	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 = \mu x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$
e^{u}	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_5 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$
u^{-2}	5	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\ (\mu = -2), \mathbf{X}_6 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}$
$u^{-\frac{2}{3}}$	5	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\ (\mu = -\frac{2}{3}), \mathbf{X}_7 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$

TABLE VII. Point symmetry classification for the class of PDEs (3.56).

$c(T)$	$c(u)$	No.	Admitted point symmetries in (X, T, U) coordinates	(x, u) components of admitted symmetries
Arbitrary	Arbitrary	3	$\mathbf{W}_1 = \frac{\partial}{\partial U}, \mathbf{W}_2 = \frac{\partial}{\partial X}$, $\mathbf{W}_3 = (X + \int^T c^2(\xi)d\xi)\frac{\partial}{\partial X} + U\frac{\partial}{\partial U}$	$\check{\mathbf{W}}_2 = \frac{\partial}{\partial x}$, $\check{\mathbf{W}}_3 = (x + \int^u c^2(\xi)d\xi)\frac{\partial}{\partial x}$
T^μ	u^μ	4	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\ (c(T) = T^\mu)$, $\mathbf{W}_4 = T\frac{\partial}{\partial T} + (2\mu+1)X\frac{\partial}{\partial X} + (\mu+1)U\frac{\partial}{\partial U}$	$\check{\mathbf{W}}_2, \check{\mathbf{W}}_3\ (c(u) = u^\mu)$, $\check{\mathbf{W}}_4 = u\frac{\partial}{\partial u} + (2\mu+1)x\frac{\partial}{\partial x}$
e^T	e^u	4	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\ (c(T) = e^T)$, $\mathbf{W}_5 = \frac{\partial}{\partial T} + 2X\frac{\partial}{\partial X} + U\frac{\partial}{\partial U}$	$\check{\mathbf{W}}_2, \check{\mathbf{W}}_3\ (c(u) = e^u)$, $\check{\mathbf{W}}_5 = \frac{\partial}{\partial u} + 2x\frac{\partial}{\partial x}$
T^{-2}	u^{-2}	5	$\mathbf{W}_1, \mathbf{W}_2$, $\mathbf{W}_3\ (c(T) = T^{-2}), \mathbf{W}_4\ (\mu = -2)$, $\mathbf{W}_6 = U^2\frac{\partial}{\partial U} + TU\frac{\partial}{\partial T} - \frac{U}{T^3}\frac{\partial}{\partial X}$	$\check{\mathbf{W}}_1, \check{\mathbf{W}}_2$, $\check{\mathbf{W}}_3\ (c(u) = u^{-2}), \check{\mathbf{W}}_4\ (\mu = -2)$, $\check{\mathbf{W}}_6 = u(t+v)\frac{\partial}{\partial u} - \frac{t+v}{u^3}\frac{\partial}{\partial x}$
$T^{-\frac{2}{3}}$	$u^{-\frac{2}{3}}$	5	$\mathbf{W}_1, \mathbf{W}_2$, $\mathbf{W}_3\ (c(T) = T^{-\frac{2}{3}}), \mathbf{W}_4\ (\mu = -\frac{2}{3})$, $\mathbf{W}_7 = (XT - 3T^{\frac{2}{3}})\frac{\partial}{\partial T} + (XT^{-\frac{1}{3}} - \frac{X^2}{3})\frac{\partial}{\partial X}$	$\check{\mathbf{W}}_1, \check{\mathbf{W}}_2$, $\check{\mathbf{W}}_3\ (c(u) = u^{-\frac{2}{3}}), \check{\mathbf{W}}_4\ (\mu = -\frac{2}{3})$, $\check{\mathbf{W}}_7 = (xu - 3u^{\frac{2}{3}})\frac{\partial}{\partial u} + (xu^{-\frac{1}{3}} - \frac{x^2}{3})\frac{\partial}{\partial x}$

The point symmetry classification for the class of PDEs (3.56), modulo its equivalence transformations (4.9), is presented in Table VII.

Remark 6. In order to determine whether a symmetry $\mathbf{W}$ of a PDE (3.56) yields a nonlocal symmetry of the corresponding nonlinear wave equation (3.50), we need to trace back to the nonlinear wave equation (3.50) using the PDE system (3.53). Since the PDE (3.56) excludes the dependent variable V of the PDE system (3.53), we need to investigate how the variable V changes under the action induced by $\mathbf{W}$. Since $\rho^{-1}(\frac{\partial}{\partial V}) = \frac{\partial}{\partial v} - \frac{\partial}{\partial t}$, where ρ^{-1} is the inverse of the transformation (3.52), the infinitesimal components for the variables x and u remain invariant when tracing back. This is why we only present the (x, u) components of admitted symmetries in Table VII.

Proposition 2. The symmetries $\mathbf{W}_6$ and $\mathbf{W}_7$ yield nonlocal symmetries of the corresponding potential systems (3.51).

Proof. If the symmetry $\mathbf{W}_6$ yields a local symmetry $\tilde{\mathbf{W}}_6$ of the potential system (3.51) with $c(u) = u^{-2}$, then, in evolutionary form, $\tilde{\mathbf{W}}_6 = \left(U^2 - TUU_T + \frac{U}{T^3}U_X\right)\frac{\partial}{\partial U} + F[U, V]\frac{\partial}{\partial V}$, where the differential function $F[U, V]$ must depend on X, T, U, V and the partial derivatives of U and V with respect to X and T. By applying $\tilde{\mathbf{W}}_6$ to the corresponding PDE system (3.53), which is invertibly related to the potential system (3.51), one can show that $F[U, V]$ must be of the form $F(X, T, U, V, U_X, U_T)$. Applying $\tilde{\mathbf{W}}_6^{(\infty)}$ to the corresponding PDE system (3.53) and making appropriate substitutions, one can prove that the resulting determining equation system is inconsistent. Hence, $\mathbf{W}_6$ yields a nonlocal symmetry of the potential system (3.51) with $c(u) = u^{-2}$.

By similar reasoning, it turns out that $\mathbf{W}_7$ also yields a nonlocal symmetry of the potential system (3.51) with $c(u) = u^{-\frac{2}{3}}$. □

When $c(u)$ is arbitrary, in (x, t, u, v) coordinates, $\mathbf{W}_3 = (x + \int^u c^2(\xi)d\xi)\frac{\partial}{\partial x} + (t + v)\frac{\partial}{\partial t}$. It is straightforward to show that $\mathbf{W}_3$ is a point symmetry of a potential system (3.51) for arbitrary $c(u)$, whose infinitesimal component for the variable t has an essential dependence on v. By projection, $\mathbf{W}_3$ yields a nonlocal symmetry of a nonlinear wave equation (3.50) for arbitrary $c(u)$.

When $c(u) = u^{-2}$, the infinitesimal components for the variables (x, u) of the symmetry $\mathbf{W}_6$ depend on the variable v. By Remark 6, $\mathbf{W}_6$ yields a nonlocal symmetry of the corresponding nonlinear wave equation (3.50).

Consider the case when $c(u) = u^{-\frac{2}{3}}$. If the symmetry $\mathbf{W}_7$ yields a local symmetry $\tilde{\mathbf{W}}_7$ of the corresponding nonlinear wave equation (3.50), then $\tilde{\mathbf{W}}_7 = \check{\mathbf{W}}_7 + f[u]\frac{\partial}{\partial t}$, where the differential function $f[u]$ depends on x, t, u and the partial derivatives of u with respect to x and t. Since $\rho^{-1}(\frac{\partial}{\partial V}) = \frac{\partial}{\partial v} - \frac{\partial}{\partial t}$, when tracing back to the corresponding potential system (3.51), the infinitesimal component for the variable v must be equal to $-f[u]$. Thus $\mathbf{W}_7$ would also yield a local symmetry of the corresponding potential system (3.51), which is a contradiction since $\mathbf{W}_7$ yields a nonlocal symmetry of the corresponding potential system (3.51). Hence $\mathbf{W}_7$ yields a nonlocal symmetry of the nonlinear wave equation (3.50) with $c(u) = u^{-\frac{2}{3}}$.

Remark 7. One can show that the symmetries $\mathbf{W}_4$ and $\mathbf{W}_5$, respectively, yield point symmetries $\tilde{\mathbf{W}}_4 = \mathbf{W}_4 + (\mu + 1)V\frac{\partial}{\partial V}$ and $\tilde{\mathbf{W}}_5 = \mathbf{W}_5 + V\frac{\partial}{\partial V}$ of the corresponding PDE systems (3.53). In terms of (x, t, u, v) coordinates, $\tilde{\mathbf{W}}_4 = u\frac{\partial}{\partial u} + (2\mu + 1)x\frac{\partial}{\partial x} + (\mu + 1)t\frac{\partial}{\partial t} + (\mu + 1)v\frac{\partial}{\partial v}$ and $\tilde{\mathbf{W}}_5 = \frac{\partial}{\partial u} + 2x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$. Hence, by projection, $\mathbf{W}_4$ and $\mathbf{W}_5$ yield point symmetries of the nonlinear wave equations (3.50) with $c(u) = u^{\mu}$ and $c(u) = e^{u}$, respectively.

Remark 8. Comparing the symmetries listed in Ref. 10, *one sees that the symmetries* $\mathbf{W}_6$ *and* $\mathbf{W}_7$ *yield previously unknown nonlocal symmetries of the nonlinear wave equations (3.50) with* $c(u) = u^{-2}$ *and* $c(u) = u^{-\frac{2}{3}}$, *respectively.*

V. CONCLUSION AND OPEN PROBLEMS

In this paper, we presented a new systematic symmetry-based procedure to construct nonlocally related PDE systems (inverse potential systems) for a given PDE system. The starting point for this method is any point symmetry of a given PDE system. Our new symmetry-based method yields previously unknown nonlocally related PDE systems for nonlinear reaction-diffusion equations, nonlinear diffusion equations, and nonlinear wave equations as well as nonlocal symmetries for nonlinear diffusion and nonlinear wave equations. Most importantly, through the symmetry-based method, we have obtained previously unknown nonlocal symmetries for nonlinear wave equations with $c(u) = u^{-2}$ or $c(u) = u^{-\frac{2}{3}}$.

Potential systems are under-determined for a given PDE system with more than two independent variables. It is known that point symmetries of such potential systems cannot yield nonlocal symmetries of the given PDE system without additional gauge constraints relating potential variables and their derivatives.[20] In the case of three or more independent variables, the inverse potential systems generated by the symmetry-based method presented in this paper involve natural gauge constraints due to their construction from curl-type conservation laws. Are there examples of such inverse potential systems, especially for given nonlinear systems of physical interest, that yield nonlocal symmetries?

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