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# On locally and nonlocally related potential systems

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For any partial differential equation (PDE) system, a local conservation law yields potential equations in terms of some potential variable, which normally is a nonlocal variable. The current paper examines situations when such a potential variable is a local variable, i.e., is a function of the independent and dependent variables of a given PDE system, and their derivatives. In the case of two independent variables, a simple necessary and sufficient condition is presented for the locality of such a potential variable, and this is illustrated by several examples. As a particular example, two-dimensional reductions of equilibrium equations for fluid and plasma dynamics are considered. It is shown that such reductions with respect to helical, axial, and translational symmetries have conservation laws which yield local potential variables. This leads to showing that the well-known Johnson–Frieman–Kruskal–Oberman (JFKO) and Bragg–Hawthorne (Grad–Shafranov) equations are locally related to the corresponding helically and axially symmetric PDE systems of fluid/plasma dynamics. For the axially symmetric case, local symmetry classifications and arising invariant solutions are compared for the original PDE system and the Bragg–Hawthorne (potential) equation. The potential equation is shown to have additional symmetries, denoted as *restricted symmetries*. Restricted symmetries leave invariant a family of solutions of a given PDE system but not the whole solution manifold, and hence are not symmetries of the given PDE system. Corresponding reductions are shown to yield solutions, which are not obtained as invariant solutions from local symmetry reduction. © *2010 American Institute of Physics.* [doi:10.1063/1.3432619]

## I. INTRODUCTION

Potentials and potential theory are widely used as tools for formulating and solving problems in mechanics, field theory, electromagnetism, fluid dynamics, etc. The introduction of auxiliary potential variables often allows one to recast a given partial differential equation (PDE) system in a form more suitable for a particular method of analysis. Potential variables and corresponding potential equations follow from local conservation laws of a PDE system. A *potential PDE system* includes a given PDE system and potential equations arising from local conservation laws.

One of the most important properties of potential variables is their nonlocality, i.e., potentials are given by nonlocal (e.g., integral) expressions in terms of the variables of a given PDE system. In general, PDE systems nonlocally related to a given one arise not only as potential systems but also as nonlocally related subsystems. A systematic procedure for construction of nonlocally related PDE systems is described in Refs. 1–3.

An important application of nonlocally related PDE systems was discovered in Refs. 4 and 5, where it was shown that potential systems can have point symmetries, which correspond to

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*nonlocal symmetries* of a given PDE system. Moreover, potential systems can lead to the systematic determination of (nonlocal) conservation laws that are not equivalent to local conservation laws of a given PDE system.[6]

In the vast subsequent literature, such nonlocal symmetries and nonlocal conservation laws have been found for many PDE systems arising in applications. For example, nonlocal symmetries were found for the nonlinear heat and wave equations,[4,7,8] the equations of planar gas dynamics,[2,9,10] the equations of nonlinear elasticity,[11] Maxwell's equations,[12] and many other PDE systems. Nonlocal symmetries have been successfully used for the construction of exact invariant solutions of nonlinear PDE systems, which do not arise as invariant solutions with respect to local symmetries (e.g., see Refs. 2, 4, and 11). Infinite-dimensional groups of nonlocal symmetries and infinite sets of nonlocal conservation laws can be used to derive a mapping of a nonlinear PDE system into an equivalent linear PDE system by a noninvertible transformation.[13,14]

Since a given PDE system and PDE systems nonlocally related to it have the same solution sets, it follows that any general method of analysis that fails to work for a given PDE system, especially a method that is not coordinate dependent, could turn out to be successful when applied to a nonlocally related system. In particular, many examples of PDE systems are known that have useful nonlocal conservation laws, noninvertible linearizations, or additional physical exact solutions, which are found through considerations of nonlocally related PDE systems. Many examples can be found in Ref. 1 and references therein.

Normally, one would expect a potential system to be able to lead to the above-described new results for a given PDE system only when its potential variables are functionally independent of the local variables of the given PDE system. This paper addresses two questions: (i) determining the conditions under which a potential variable is functionally dependent on local variables and (ii) determining restricted symmetries arising from such "local" potential variables that are not local symmetries of the given PDE system. For a given PDE system, topologically, such restricted symmetries leave invariant a family of its solutions but do not leave invariant the whole solution manifold.

We only consider PDE systems with two independent variables. The paper is organized as follows.

In Sec. II, we present a simple necessary and sufficient condition for a potential variable to be a local variable and give basic examples.

In Sec. III, we consider the main physical example: the time-independent PDE system of incompressible Euler equations of fluid dynamics in three space dimensions [or, equivalently, magnetohydrodynamics (MHD) equations describing static equilibria of ideal plasmas] and its two-dimensional helically, axially, and translationally symmetric versions. These PDE systems are widely used in applications. In particular, helical fluid flows are known to form under various conditions (e.g., Ref. 15). Helically symmetric dynamic and equilibrium plasma configurations are important in plasma confinement, in particular, in tokamak theory, as well as in astrophysical modeling (e.g., Refs. 16–18). Axially symmetric MHD equilibrium equations have been used to derive families of exact plasma equilibria.[19–21]

Using a conservation law (incompressibility condition) for each of the three two-dimensional reductions of Euler equations, a potential variable (the flux function) is introduced. Then in each of the reductions, the corresponding Euler system can be written as a single equation for the flux function, yielding the fundamental equations of fluid and plasma theory: the Johnson–Frieman–Kruskal–Oberman (JFKO) equation[22] (helical symmetry), the Bragg–Hawthorne (Grad–Shafranov) equation[23–26] (axial symmetry), and the corresponding PDE for the translational symmetry. Bragg–Hawthorne and JFKO equations are widely used in fluid and plasma modeling.

In each symmetry reduction, the potential variable (flux function) is functionally dependent on local variables of the problem and, in particular, this specific dependence plays the role of a constitutive function.

In Sec. IV, we compare the Lie point symmetries of the axially symmetric MHD equilibrium equations and the Bragg–Hawthorne equation for the potential variable. Interestingly, even though the two PDE systems have a local relationship, their point symmetry classifications are rather different. We study the seemingly paradoxical relationship between these point symmetry classifications. In particular, we show that the Bragg–Hawthorne equation has point symmetries (including an infinite number of point symmetries for the linear case), which are *restricted symmetries* of the original axially symmetric system of Euler equations since they turn out to only hold for a particular class of solutions.

Finally, in Sec. V, we compare invariant solutions of the axially symmetric MHD equilibrium equations with classes of solutions invariant with respect to restricted symmetries of the Bragg–Hawthorne equation. We show that in several cases, the consideration of restricted symmetries of the Bragg–Hawthorne equation yields additional solutions, which are not invariant with respect to any point symmetries of the axially symmetric Euler equations. This example illustrates that considering a potential formulation for symmetry analysis can lead to obtaining new solutions, even in the case when potential variables are local variables.

The symbolic software package GEM for MAPLE (Ref. 27) was used for all symmetry computations.

## II. CONDITIONS FOR THE LOCALITY OF A POTENTIAL VARIABLE

### A. Conservation laws and potential systems

Consider a PDE system $\mathbf{R}\{x,t;u\}$ of order $k$, with $m$ dependent variables $u=(u^1,\ldots,u^m)$ and two independent variables $(x,t)$,

$$R^\sigma[u]=R^\sigma(x,t,u,\partial u,\ldots,\partial^k u)=0,\quad \sigma=1,\ldots,N. \tag{2.1}$$

Here, $\partial u$ denotes first-order partial derivatives and $\partial^p u$ denotes $p$th order partial derivatives appearing in (2.1), $1\le p\le k$.

A local conservation law of the PDE system (2.1) is given by

$$\mathrm{D}_t\Psi[u]+\mathrm{D}_x\Phi[u]=0 \tag{2.2}$$

for some density $\Psi[u]=\Psi(x,t,u,\partial u,\ldots,\partial^r u)$ and flux $\Phi[u]=\Phi(x,t,u,\partial u,\ldots,\partial^r u)$, $r\ge 0$. In Eq. (2.2), the total derivative operators are given by

$$\mathrm{D}_t=\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{xt}\frac{\partial}{\partial u_x}+u_{tt}\frac{\partial}{\partial u_t}+\ \ldots\ ,$$

$$\mathrm{D}_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+\ \ldots\ .$$

For any given PDE system (2.1), local conservation laws (2.2) can be systematically sought using the direct method[28,29] involving multipliers.

Each conservation law (2.2) yields a pair of *potential equations*

$$\mathcal{P}\colon\begin{cases}v_x=\Psi[u]\\ v_t=-\Phi[u]\end{cases} \tag{2.3}$$

for some auxiliary *potential variable* (potential) $v=v(x,t)$. A *potential system* $\mathbf{S}\{x,t;u,v\}$ is given by the union of the given system (2.1) and the potential equation (2.3),

$$\mathbf{S}\{x,t;u,v\}\colon\begin{cases}R^\sigma[u]=R^\sigma(x,t,u,\partial u,\ldots,\partial^k u)=0,\quad \sigma=1,\ldots,N\\ v_x=\Psi[u]\\ v_t=-\Phi[u].\end{cases} \tag{2.4}$$

[Redundant equations can be excluded from (2.4) or kept.]

The potential system $\mathbf{S}\{x,t;u,v\}$ (2.4) has essentially the same solution set as that of the given PDE system $\mathbf{R}\{x,t;u\}$ (2.1). In particular, if $u=\Theta(x,t)$ is a solution of (2.1), then due to the satisfaction of the integrability condition $v_{xt}=v_{tx}$, it follows that there is a corresponding solution $v=\Gamma(x,t)$ of the potential system (2.4) unique to within an arbitrary constant, i.e., if $(u,v)=(\Theta(x,t),\Gamma(x,t))$ is a solution of the potential system (2.4), then so is $(u,v)=(\Theta(x,t),\Gamma(x,t)+C)$ for any constant $C$. Conversely, if $(u,v)=(\Theta(x,t),\Gamma(x,t))$ solves the potential system (2.4), then by projection, $u=\Theta(x,t)$ solves the given PDE system (2.1). Consequently, through this relationship between their solution sets, the potential system $\mathbf{S}\{x,t;u,v\}$ (2.4) is nonlocally equivalent to the given PDE system $\mathbf{R}\{x,t;u\}$ (2.1), and the mapping that relates systems (2.1) and (2.4) is noninvertible.

### B. The condition for locality of a potential variable

For the case of two independent variables, it is straightforward to establish a necessary and sufficient condition for a potential to be a local variable. Indeed, the following lemma holds.

*Lemma 1: Suppose a PDE system* $\mathbf{R}\{x,t;u\}$ *(2.1) has a local conservation law (2.2). The corresponding potential variable* $v(x,t)$ *defined by the potential equation (2.3) is a local variable [i.e., for the general solution of (2.4),* $v(x,t)$ *is a function of at most* $x,t,u(x,t)$ *and partial derivatives of* $u(x,t)$*] if and only if there exists a function* $g[u]=g(x,t,u,\partial u,\ldots,\partial^l u)$, $l\geq 0$, *such that the equation*

$$\Psi[u]\mathrm{D}_t g[u]+\Phi[u]\mathrm{D}_x g[u]=0 \tag{2.5}$$

*holds on solutions* $u(x,t)$ *of the PDE system (2.1).*

*Proof:* It is well known that if, for two smooth functions $f(x,t)$ and $h(x,t)$ defined on an open neighborhood of some point $(x_0,t_0)$, the Poisson bracket

$$\{f(x,t),h(x,t)\}_{\{x,t\}}\equiv f_x h_t - f_t h_x = 0,$$

and $|\mathrm{grad}\, h(x,t)|\neq 0$, then $f(x,t)$ and $h(x,t)$ are functionally dependent, in particular, there exists a function $F(s)$ such that $f(x,t)=F(h(x,t))$.

On solutions $u(x,t)$ of the given PDE system $\mathbf{R}\{x,t;u\}$ (2.1), the potential variable $v(x,t)$ is functionally dependent on the variables of $\mathbf{R}\{x,t;u\}$ if and only if there is some function $g[u]=g(x,t,u,\partial u,\ldots,\partial^l u)$, $l\geq 0$, such that

$$\{v(x,t),g[u]\}_{\{x,t\}}=v_x\mathrm{D}_t g[u]-v_t\mathrm{D}_x g[u]=0. \tag{2.6}$$

Substituting the potential equation (2.3) into (2.6), one obtains the statement of the lemma. □

The simplest example is given by the linear advection equation

$$u_t+u_x=0. \tag{2.7}$$

The solution of (2.7) is obviously given by $u(x,t)=G(x-t)$, where $G$ is an arbitrary smooth function. Equation (2.7) is a conservation law as it stands. The corresponding potential equation (2.3) is given by

$$v_x=u,$$

$$v_t=-u. \tag{2.8}$$

Using, for example, $g(u)=u$ in (2.5), one obtains

$$u\mathrm{D}_t u+u\mathrm{D}_x u=u(u_t+u_x)=0$$

on all solutions of (2.7). Hence, the potential $v$ is functionally dependent on $u$. Another way to see this dependence is to observe that from (2.8), $v_t+v_x=0$; hence, $v(x,t)=H(x-t)$, and therefore $v$ is functionally dependent on $u(x,t)=G(x-t)$.

As a second example, consider the PDE system $\mathbf{UW}\{x,t;u,w\}$ given by

$$u_t = w_{xx},$$

$$w_t = -u^{-1}(u-w)_x w_x + w_{xx}. \tag{2.9}$$

The first equation of (2.9) is a conservation law as it stands. Hence, one can introduce a potential variable $v$, defined by the potential equations

$$v_x = u,$$

$$v_t = w_x. \tag{2.10}$$

Let $g[u,w]=u-w$. Then (2.5) becomes

$$v_x(u_t - w_t) - v_t(u_x - w_x) = u(u^{-1}(u-w)_x w_x) - w_x(u_x - w_x) = 0$$

on solutions of (2.10). It follows that the potential variable $v$ is functionally dependent on the linear combination $u-w$, and hence is a local variable of the PDE system (2.9).

## III. LOCAL POTENTIALS IN TWO-DIMENSIONAL REDUCTIONS OF FLUID AND PLASMA EQUILIBRIUM EQUATIONS

### A. Equations of fluid and plasma equilibria in three dimensions

The well-known Euler system of fluid dynamics equations in three dimensions, describing inviscid incompressible flows, is given by

$$\operatorname{div} \mathbf{V} = 0, \quad \mathbf{V}_t + (\mathbf{V} \cdot \operatorname{grad})\mathbf{V} = -\frac{1}{\rho}\operatorname{grad} p. \tag{3.1}$$

Here, $\mathbf{V}=V^1\mathbf{e}_x+V^2\mathbf{e}_y+V^3\mathbf{e}_z$ is the fluid velocity vector, $p$ is the fluid pressure, and $\rho$=const is the fluid density. (Throughout this paper, we use upper index notation for components of vector fields.)

The equilibrium version of the Euler equation (3.1) is given by the PDE system

$$\operatorname{div} \mathbf{V} = 0, \quad \mathbf{V} \times (\operatorname{curl} \mathbf{V}) = \operatorname{grad}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{V}|^2\right). \tag{3.2}$$

The nonlinear PDE system (3.2) includes four equations for four independent variables.

Interestingly, the same PDE system arises in a completely different application. In the ideal MHD framework, the PDE system describing static plasma equilibrium configurations is given by the system $\mathbf{BP}\{x,y,z;B^1,B^2,B^3,P\}$,

$$\operatorname{div} \mathbf{B} = 0, \quad (\operatorname{curl} \mathbf{B}) \times \mathbf{B} = \operatorname{grad} P, \tag{3.3}$$

where $\mathbf{B}=B^1\mathbf{e}_x+B^2\mathbf{e}_y+B^3\mathbf{e}_z$ is the magnetic field vector and $P$ is the plasma pressure. Through the association $\mathbf{V}=\mathbf{B}$ and $p/\rho+|\mathbf{V}|^2/2=P_0-P$, $P_0$=const, one observes that systems (3.2) and (3.3) coincide. From now on, we consider only the MHD equilibrium system $\mathbf{BP}\{x,y,z;B^1,B^2,B^3,P\}$ (3.3).

Since $\mathbf{B}\cdot\operatorname{grad} P=0$, the magnetic field lines lie on *magnetic surfaces*, which are the level surfaces $P(x,y,z)$=const. For bounded plasma configurations without edges, if either $\mathbf{B}$ or curl $\mathbf{B}$ nowhere vanishes in the plasma domain, it follows that such magnetic surfaces are nested tori.[30]

The PDE system (3.3) is obviously invariant under spatial translations and rotations. In particular, in terms of Lie point symmetries, the system $\mathbf{BP}\{x,y,z;B^1,B^2,B^3,P\}$ (3.3) has the translation symmetry in the $z$-direction with infinitesimal generator

$$\mathrm{X}_T = \frac{\partial}{\partial z}, \tag{3.4}$$

and the axial symmetry with infinitesimal generator

$$\mathrm{X}_R = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \phi} \tag{3.5}$$

($\phi$ is the polar angle). In general, the PDE system (3.3) has the *helical symmetry* corresponding to any linear combination of the infinitesimal generators $\mathrm{X}_T$ and $\mathrm{X}_R$ given by

$$\mathrm{X}_H = a\frac{\partial}{\partial \phi} - b\frac{\partial}{\partial z}, \tag{3.6}$$

where the case $ab \neq 0$ corresponds to a genuine helical symmetry, and the special cases $a = 0,\ b \neq 0$ and $a \neq 0,\ b = 0$ to translation and axial symmetries (3.4) and (3.6), respectively.

### B. The three two-dimensional reductions

We now consider the two-dimensional reductions of the MHD equilibrium system $\mathbf{BP}\{x,y,z;B^1,B^2,B^3,P\}$ (3.3) with respect to the point symmetries (3.4)–(3.6). Here, it is natural to rewrite the PDE system (3.3) in terms of cylindrical coordinates $(r,\phi,z)$,

$$\mathbf{B} = B^1(r,\phi,z)\mathbf{e}_r + B^2(r,\phi,z)\mathbf{e}_\phi + B^3(r,\phi,z)\mathbf{e}_z, \quad P = P(r,\phi,z). \tag{3.7}$$

It will be seen that reductions with respect to the three symmetries (3.4)–(3.6) will have similar forms. Hence, we proceed with the reduction with respect to the general symmetry (3.6), $a^2+b^2 > 0$.

#### 1. The general helically symmetric reduction

Choosing canonical coordinates for the helical symmetry (3.6), note that one can choose the polar radius $r$ and the quantity $u = az + b\phi$ as the two invariants. As the third canonical coordinate, we choose $v = a\phi - bz/r^2$, which runs along each helix. The corresponding invariant physical variables in the coordinates $(r,u,v)$ take the form

$$\mathbf{B} = B^1(r,u)\mathbf{e}_r + B^u(r,u)\mathbf{e}_u + B^v(r,u)\mathbf{e}_v, \quad P = P(r,u),$$

where

$$B^u(r,u) = \frac{b}{r}B^2(r,\phi,z) + aB^3(r,\phi,z) \quad \text{and} \quad B^v(r,u) = aB^2(r,\phi,z) - \frac{b}{r}B^3(r,\phi,z).$$

In this notation, system (3.3) can be written as the PDE system $\mathbf{H}\{r,u;B^1,B^u,B^v,P\}$ given by

$$(rB^1)_r + (rB^u)_u = 0, \tag{3.8a}$$

$$B^u(B^1)_u - \tfrac{1}{2}r^2M(r)[(B^u)^2 + (B^v)^2]_r - rM^2(r)(b^2[(B^u)^2 + (B^v)^2] + [bB^u + arB^v]^2) = P_r, \tag{3.8b}$$

$$B^1(rB^v)_r + B^u(rB^v)_u = 0, \tag{3.8c}$$

$$-B^1(B^1)_u + r^2M(r)[B^1(B^u)_r - B^v(B^v)_u] + 2brM^2(r)B^1[bB^u + arB^v] = P_u, \tag{3.8d}$$

where

$$M(r) = (a^2r^2 + b^2)^{-1}.$$

Equation (3.8a) is a conservation law as it stands with fluxes

$$\Phi^r = rB^1, \quad \Phi^u = rB^u, \tag{3.9}$$

and hence leads to potential variable $\psi(r,u)$ satisfying the potential equations

$$\psi_u = rB^1,$$

$$\psi_r = -\, rB^u. \tag{3.10}$$

The corresponding potential system $\mathbf{H\Psi}\{r,u;B^1,B^u,B^v,P,\psi\}$ is given by

$$\psi_u = rB^1, \tag{3.11a}$$

$$\psi_r = -\, rB^u, \tag{3.11b}$$

$$B^u(B^1)_u - \tfrac{1}{2}r^2M(r)[(B^u)^2 + (B^v)^2]_r - rM^2(r)(b^2[(B^u)^2 + (B^v)^2] + [bB^u + arB^v]^2) = P_r, \tag{3.11c}$$

$$B^1(rB^v)_r + B^u(rB^v)_u = 0, \tag{3.11d}$$

$$-\, B^1(B^1)_u + r^2M(r)[B^1(B^u)_r - B^v(B^v)_u] + 2brM^2(r)B^1[bB^u + arB^v] = P_u. \tag{3.11e}$$

Multiplying Eq. (3.11c) by $r$, one obtains

$$\Phi^r(rB^v)_r + \Phi^u(rB^v)_u = 0,$$

where $\Phi^r$ and $\Phi^u$ are given by (3.9). From Lemma 1 with $g=rB^v$, it follows that the potential variable $\psi(r,z)$ is a *local variable*, in particular,

$$\psi = F(rB^v) \quad \text{or} \quad B^v = \frac{I(\psi)}{r},$$

for any functions $F$ and $I$ of their respective arguments. Consequently, the potential system $\mathbf{H\Psi}\{r,u;B^1,B^u,B^v,P,\psi\}$ (3.11) is *locally related* to the helically symmetric MHD equilibrium system $\mathbf{H}\{r,u;B^1,B^u,B^v,P\}$ (3.8). Moreover, using Eqs. (3.8b) and (3.8d), one finds that the Poisson bracket

$$\{\psi,P\}_{\{r,u\}} = -\,(b^2 + a^2r^2)I(\psi)\{\psi,I(\psi)\}_{\{r,u\}} = 0,$$

and hence the pressure $P=P(\psi)$. Subsequently, from (3.8b) and (3.8d), one finds that both the helically symmetric PDE system (3.8) and its potential system (3.11) are locally equivalent to the scalar equation (*the JFKO equation*[22])

$$r^2M(r)\psi_{rr} - rM^2(r)(a^2r^2 - b^2)\psi_r + \psi_{uu} + r^2M(r)I(\psi)I'(\psi) - 2abr^2M^2(r)I(\psi) = -\, r^2P'(\psi) \tag{3.12}$$

for the unknown function $\psi(r,u)$, where $I(\psi)$ and $P(\psi)$ can be treated as arbitrary smooth functions (constitutive functions). The corresponding physical variables are given by

$$\mathbf{B} = \frac{\psi_z}{r}\mathbf{e}_r + M(r)[arI(\psi) - b\psi_r]\mathbf{e}_\phi - M(r)[ar\psi_r + bI(\psi)]\mathbf{e}_z, \quad P = P(\psi).$$

Here, it is essential to note that the solutions of (3.12) for a particular choice of $I(\psi)$ and $P(\psi)$

correspond to a *subclass* of solutions of the PDE system (3.8) or its potential system (3.11).

Physically, since $P=P(\psi)$, the level surfaces $\psi(r,u)=\text{const}$ define magnetic surfaces of the helically symmetric plasma configuration.

We now single out two important cases included in the above derivation that are often used independently in applications.

#### 2. The axially symmetric reduction

To consider the rotationally invariant version of the MHD equilibrium PDE system (3.3), one sets $a=1$, $b=0$, $u=z$, and $v=\phi$ in the above derivation. Then the physical variables take on the form

$$\mathbf{B}=B^1(r,z)\mathbf{e}_r+B^2(r,z)\mathbf{e}_\phi+B^3(r,z)\mathbf{e}_z,\quad P=P(r,z). \tag{3.13}$$

The corresponding rotationally invariant PDE system of MHD equilibrium equations $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ obtained from (3.8) is given by the four equations

$$(rB^1)_r+(rB^3)_z=0, \tag{3.14a}$$

$$rB^3[(B^1)_z-(B^3)_r]-B^2(rB^2)_r=rP_r, \tag{3.14b}$$

$$B^1(rB^2)_r+B^3(rB^2)_z=0, \tag{3.14c}$$

$$-B^2(B^2)_z+B^1[(B^3)_r-(B^1)_z]=P_z. \tag{3.14d}$$

The corresponding locally related potential system $\mathbf{A\Psi}\{r,z;B^1,B^2,B^3,P,\psi\}$ is given by

$$\begin{gathered}\psi_z=rB^1,\\ \psi_r=-rB^3,\\ rB^3[(B^1)_z-(B^3)_r]-B^2(rB^2)_r=rP_r,\\ B^1(rB^2)_r+B^3(rB^2)_z=0,\\ -B^2(B^2)_z+B^1[(B^3)_r-(B^1)_z]=P_z,\end{gathered} \tag{3.15}$$

where $\psi=F(rB^2)$ or $B^2=I(\psi)/r$, and $P=P(\psi)$. Thus, both the PDE system (3.14) and the potential system (3.15) are locally equivalent to the well-known scalar Bragg–Hawthorne (Grad–Shafranov) equation $\mathbf{\Psi}\{r,z;\psi\}$,[23–26]

$$\psi_{rr}-\frac{1}{r}\psi_r+\psi_{zz}+I(\psi)I'(\psi)=-r^2P'(\psi), \tag{3.16}$$

where $I(\psi)$ and $P(\psi)$ are arbitrary constitutive functions. (It is interesting to note that the Bragg–Hawthorne equation has been rediscovered many times. After its original derivation in the context of fluid dynamics in 1950,[23] it was found by Lüst and Schlüter in plasma physics in 1957,[24] and then independently in 1958 by Grad and Rubin[25] and Shafranov.[26] In plasma physics, the common name of the equation is the Grad–Shafranov equation, and in fluid dynamics, it is commonly referred to as the Bragg–Hawthorne equation.) The magnetic field and pressure are given by

$$\mathbf{B}=\frac{\psi_z}{r}\mathbf{e}_r+\frac{I(\psi)}{r}\mathbf{e}_\phi-\frac{\psi_r}{r}\mathbf{e}_z,\quad P=P(\psi). \tag{3.17}$$

#### *3. The translationally symmetric reduction*

The version of the MHD equilibrium PDE system (3.3) invariant with respect to translations (3.4) is obtained by setting $a=0$, $b=1$ in Sec. III B 1. However, it is more convenient to use the Cartesian representation $\mathbf{B}=B^x(x,y)\mathbf{e}_x+B^y(x,y)\mathbf{e}_y+B^z(x,y)\mathbf{e}_z$, $P=P(x,y)$. It follows that these quantities satisfy the PDE system $\mathbf{T}\{x,y;B^x,B^y,B^z,P\}$ given by

$$(B^x)_x+(B^y)_y=0, \tag{3.18a}$$

$$-B^3(B^z)_x-B^2[(B^y)_x-(B^x)_y]=P_x, \tag{3.18b}$$

$$-B^3(B^z)_y-B^1[(B^x)_y-(B^y)_x]=P_y, \tag{3.18c}$$

$$B^2(B^z)_y-B^1(B^z)_x=0. \tag{3.18d}$$

One uses the conservation law (3.18a) to introduce a potential variable $\xi(x,y)$ satisfying

$$\xi_x=B^y,$$

$$\xi_y=-B^x \tag{3.19}$$

From Eq. (3.18d) and Lemma 1 with $g=B^z$, it follows that the potential variable $\xi$ is a local variable,

$$\xi=F(B^z) \quad \text{or} \quad B^z=I(\xi),$$

and moreover, $P=P(\xi)$. Hence, it is easy to show that the PDE system (3.18) reduces to the scalar equation

$$\xi_{xx}+\xi_{yy}=Q(\xi)\coloneqq -I(\xi)I'(\xi)-P'(\xi), \tag{3.20}$$

which is locally equivalent to the PDE system $\mathbf{T}\{x,y;B^x,B^y,B^z,P\}$ (3.18) for arbitrary $Q(\xi)$.

## IV. COMPARISON OF POINT SYMMETRIES OF LOCALLY RELATED PDE SYSTEMS (3.14)–(3.16)

In spite of equivalence and the local relations connecting the axially symmetric MHD equilibrium system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14), the potential system $\mathbf{A\Psi}\{r,z;B^1,B^2,B^3,P,\psi\}$ (3.15), and the Bragg–Hawthorne equation $\mathbf{\Psi}\{r,z;\psi\}$ (3.16), the forms of these nonlinear equations are rather different. In particular, the scalar Bragg–Hawthorne equation (3.16) has two arbitrary constitutive functions $I(\psi)$ and $P(\psi)$, whereas systems (3.14) and (3.15) have none. A comparison is now made of the Lie point symmetry structures of the PDE systems $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14), $\mathbf{A\Psi}\{r,z;B^1,B^2,B^3,P,\psi\}$ (3.15), and $\mathbf{\Psi}\{r,z;\psi\}$ (3.16).

In this section, we focus on symmetry comparisons for the axially invariant reduction of the MHD equilibrium system (3.3); the helically and translationally symmetric cases are conceptually the same, and symmetry analysis proceeds in a similar manner.

The point symmetry analysis of the PDE systems (3.14) and (3.15) yields the following results.

### A. Point symmetries of the potential system $\mathbf{A\Psi}\{r,z;B^1,B^2,B^3,P,\psi\}$ (3.15)

The potential system $\mathbf{A\Psi}\{r,z;B^1,B^2,B^3,P,\psi\}$ (3.15) has six point symmetries (three translations and three scalings),

$$\mathrm{X}_1=\frac{\partial}{\partial z}, \quad \mathrm{X}_2=\frac{\partial}{\partial P}, \quad \mathrm{X}_3=\frac{1}{r^2B^2}\frac{\partial}{\partial B^2}, \quad \mathrm{X}_4=r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}+2\psi\frac{\partial}{\partial \psi},$$

$$\mathrm{X}_5 = B^1\frac{\partial}{\partial B^1} + B^2\frac{\partial}{\partial B^2} + B^3\frac{\partial}{\partial B^3} + 2P\frac{\partial}{\partial P} + \psi\frac{\partial}{\partial \psi}, \quad \mathrm{X}_6 = \frac{\partial}{\partial \psi}. \tag{4.1}$$

### B. Point symmetries of the MHD equilibrium system A$\{r,z;B^1,B^2,B^3,P\}$ (3.14)

The axially symmetric MHD equilibrium system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) has five point symmetries given by the projections of $\mathrm{X}_1,\ldots,\mathrm{X}_5$ on the space of variables $r,z,B^1,B^2,B^3,P$,

$$\mathrm{Y}_1 = \frac{\partial}{\partial z}, \quad \mathrm{Y}_2 = \frac{\partial}{\partial P}, \quad \mathrm{Y}_3 = \frac{1}{r^2B^2}\frac{\partial}{\partial B^2}, \quad \mathrm{Y}_4 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z},$$

$$\mathrm{Y}_5 = B^1\frac{\partial}{\partial B^1} + B^2\frac{\partial}{\partial B^2} + B^3\frac{\partial}{\partial B^3} + 2P\frac{\partial}{\partial P}. \tag{4.2}$$

### C. Point symmetries of the scalar Bragg–Hawthorne equation $\Psi\{r,z;\psi\}$ (3.16)

In classifying the point symmetries of the Bragg–Hawthorne equation (3.16), first note that its equivalence transformations are given by

$$\tilde{r} = c_4c_5^{-1}r, \quad \tilde{z} = c_4c_5^{-1}z + c_1,$$

$$\tilde{\psi} = c_4^4c_5^{-2}\psi - \tfrac{1}{8}c_2r^4 - \tfrac{1}{4}c_3r^2(2\log r - 1),$$

$$\tilde{P}'(\tilde{\psi}) = c_5^2P'(\psi) + c_2, \quad \tilde{I}(\tilde{\psi})\tilde{I}'(\tilde{\psi}) = c_4^2I(\psi)I'(\psi) + c_3, \tag{4.3}$$

the pressure translation,

$$\tilde{P}(\psi) = P(\psi) + c_6, \tag{4.4}$$

as well as the well-known transformation,

$$\tilde{I}(\psi) = \pm\sqrt{I^2(\psi) + c_7}, \tag{4.5}$$

where $c_1,\ldots,c_7$ are arbitrary constants, $c_4c_5 \neq 0$.

Modulo the equivalence transformations (4.3)–(4.5), the point symmetry classification of (3.16) is presented in Table I. (Partial results for the symmetry classification given in Table I appeared in Refs. 31 and 32.)

#### *1. Relations between the point symmetries*

The point symmetry classifications for the PDE system (3.14) and the scalar Bragg–Hawthorne equation (3.16) are clearly different. We now consider the symmetry relations in detail.

For any particular choice of the constitutive functions $I(\psi)$ and $P(\psi)$, a solution of the PDE (3.16) yields a solution of system (3.14). Conversely, any solution $(B^1,B^2,B^3,P)$ of the axially symmetric PDE system (3.14) yields a solution of the Bragg–Hawthorne equation (3.16) for some particular $I(\psi)$ and $P(\psi)$; however, two different solutions $(B^1,B^2,B^3,P)$ and $(\tilde{B}^1,\tilde{B}^2,\tilde{B}^3,\tilde{P})$ of (3.14), in general, yield solutions of the PDE (3.16) corresponding to different pairs of constitutive functions $(I(\psi),P(\psi))$, $(\tilde{I}(\psi),\tilde{P}(\psi))$. In other words, the solution set of the Bragg–Hawthorne equation (3.16) for a prescribed pair of functions $(I(\psi),P(\psi))$ corresponds to a *subset* of the solution set of the PDE system (3.14).

Consequently, a symmetry of the Bragg–Hawthorne equation (3.16) leaves invariant a subset of the solution set of the PDE system (3.14), i.e., it leaves invariant a submanifold of the solution

TABLE I. Classification of point symmetries of the Bragg–Hawthorne equation (3.16).

| Case No. | Conditions on $I(\psi), P(\psi)$ | Point symmetries |
|---|---|---|
| 1 | Arbitrary | $Z_1=\dfrac{\partial}{\partial z}$. |
| 2 | $I(\psi)I'(\psi)=e^{\psi}$, $P'(\psi)=e^{2\psi}$ | $Z_1$, $Z_2=r\dfrac{\partial}{\partial r}+z\dfrac{\partial}{\partial z}-2\dfrac{\partial}{\partial\psi}$. |
| 3 | $I(\psi)I'(\psi)=\psi^{1+1/\gamma}$, $P'(\psi)=\psi^{1+2/\gamma}$, $\gamma=\text{const}$ | $Z_1$, $Z_3=r\dfrac{\partial}{\partial r}+z\dfrac{\partial}{\partial z}-2\gamma\psi\dfrac{\partial}{\partial\psi}$. |
| 4 | $I(\psi)I'(\psi)=\psi^{-3}$, $P'(\psi)=\psi^{-7}$, | $Z_1$, $Z_3(\gamma=-1/4)$, $Z_4=rz\dfrac{\partial}{\partial r}+\dfrac{1}{2}(z^2-r^2)\dfrac{\partial}{\partial z}+\dfrac{1}{2}z\psi\dfrac{\partial}{\partial\psi}$. |
| 5 | $I(\psi)I'(\psi)=a_1\psi+a_2$, $P'(\psi)=a_3\psi+a_4$ $(a_1,\ldots,a_4=\text{const})$ | $Z_1$, $Z_\infty=g\dfrac{\partial}{\partial\psi}$, $g=g(r,z)$ is an arbitrary solution of the PDE $g_{rr}-\dfrac{1}{r}g_r+g_{zz}+a_1g=-a_3r^2g$ [due to the linearity of (3.16)]. |
| 5a | $a_2=a_4=0$ | $Z_1,Z_\infty,Z_5=\psi\dfrac{\partial}{\partial\psi}$. |
| 5b | $a_1=a_2=a_3=0$ | $Z_1,Z_\infty,Z_6=r\dfrac{\partial}{\partial r}+z\dfrac{\partial}{\partial z}+4\psi\dfrac{\partial}{\partial\psi}$. |
| 5c | $a_1=a_3=a_4=0$ | $Z_1,Z_\infty,Z_7=r\dfrac{\partial}{\partial r}+z\dfrac{\partial}{\partial z}+2\psi\partial/\partial\psi$. |
| 5d | $a_1=a_2=a_3=a_4=0$ | $Z_1,Z_\infty,Z_4,Z_5,Z_6$. |

manifold of the PDE system. In particular, it leaves invariant the subset of solutions corresponding to a fixed choice of the constitutive functions $I(\psi)$ and $P(\psi)$, and hence does not necessarily yield a symmetry of system (3.14).

In particular, the differences in the point symmetry classifications of the locally related systems $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) and $\mathbf{\Psi}\{r,z;\psi\}$ (3.16) arise due to the following main reasons.

- The relation between quantities $B^2$ and $P$ is implicit in $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14). In the point symmetry analysis of the PDE system (3.14), $B^2$ and $P$ are treated as distinct dependent variables. On the other hand, in equation $\mathbf{\Psi}\{r,z;\psi\}$ (3.16), $B^2=B^2(\psi)/r$. Moreover, in the Bragg–Hawthorne equation (3.16), both $B^2=I(\psi)/r$ and $P=P(\psi)$ are not dependent variables but are treated as fixed constitutive functions. Hence, the solutions sets of (3.14) and (3.16) are not necessarily equivalent.
- The local potential variable is present in the system $\mathbf{\Psi}\{r,z;\psi\}$ (3.16) and absent in the system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14).

Firstly, we consider how the point symmetries of the Bragg–Hawthorne equation (3.16) in Table I correspond to the point symmetries (4.2) of the PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14). We consider the restricted situation when $B^2=F(P)/r$ in the PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14). Then system (3.14) reduces to the system of three PDEs $\tilde{\mathbf{A}}\{r,z;B^1,B^3,P\}$ given by

$$(rB^1)_r+(rB^3)_z=0,$$

$$r^2B^3[(B^1)_z-(B^3)_r]-F(P)F'(P)P_r=r^2P_r,$$

TABLE II. Classification of point symmetries of the PDE system (4.6).

| Case No. | Conditions on $F(P)$ | Point symmetries |
|---|---|---|
| 1 | Arbitrary | $\mathrm{N}_1=\frac{\partial}{\partial z}$. |
| 2 | $F(P)F'(P)=e^P$ | $\mathrm{N}_1$, $\mathrm{N}_2=r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}+2\frac{\partial}{\partial P}$. |
| 3 | $F(P)F'(P)=P^\alpha$ | $\mathrm{N}_1$, $\mathrm{N}_3=\alpha\left(r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}\right)+2P\frac{\partial}{\partial P}+B^1\frac{\partial}{\partial B^1}+B^3\frac{\partial}{\partial B^3}$. |
| 4 | $F'(P)=0$ | $\mathrm{N}_1$, $\mathrm{N}_3(\alpha=0)$, $\mathrm{N}_4=\frac{\partial}{\partial P}$. |
| 5 | $F(P)=0$ | $\mathrm{N}_1$, $\mathrm{N}_2$, $\mathrm{N}_3(\alpha=0)$, $\mathrm{N}_4$. |

$$B^1P_r+B^3P_z=0. \tag{4.6}$$

Here, the fourth equation of (3.14) becomes redundant. The point symmetry classification of the system $\widetilde{\mathbf{A}}\{r,z;B^1,B^3,P\}$ (4.6) with respect to its constitutive function $F(P)$ is given in Table II.

Note that cases 1–3 in Table I arise in Table II. However, the symmetries $\mathrm{Z}_4,\ldots,\mathrm{Z}_7$ and the infinite set of symmetries $\mathrm{Z}_\infty$ appearing in Table I do not correspond to point symmetries of $\widetilde{\mathbf{A}}\{r,z;B^1,B^3,P\}$ (4.6). More generally, one can show that these point symmetries do not correspond to other local (i.e., higher order) symmetries of (4.6). This means that the symmetries $\mathrm{Z}_4,\ldots,\mathrm{Z}_7$ and $\mathrm{Z}_\infty$ essentially depend on the relationship connecting the local variable $\psi$ introduced in the Bragg–Hawthorne equation (3.16) with the dependent variables $B^1$ and $B^3$ in the PDE system (3.14).

In particular, symmetries $\mathrm{Z}_4,\ldots,\mathrm{Z}_7$ and $\mathrm{Z}_\infty$ are *restricted symmetries* of the PDE system $\widetilde{\mathbf{A}}\{r,z;B^1,B^3,P\}$ (4.6), i.e., symmetries holding only for submanifolds of solutions of (4.6).

Moreover, all symmetries appearing in Table II, except for $\mathrm{N}_1$, are restricted symmetries of the PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) since the latter system is related to the PDE system (4.6) through the "restriction" $B^2=F(P)/r$, with special cases of this restriction listed in Table II. Furthermore, the symmetries $\mathrm{Z}_4$ and $\mathrm{Z}_\infty$ appear under even more restricted conditions that cannot be formulated in terms of a single constitutive function $F(P)$.

Secondly, we consider the correspondence between the point symmetries (4.2) of the PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) and the point symmetries of the Bragg–Hawthorne equation (3.16). Due to the correspondence $P(r,z)=P(\psi)$, $B^2(r,z)=I(\psi)/r$, it is natural to seek analogs of the point symmetries (4.2) through the equivalence transformations (4.3)–(4.5) of the Bragg–Hawthorne equation (3.16). The following relations are observed.

- The point symmetry $\mathrm{Y}_1$ corresponds to the equivalence transformation (4.3) with $c_1$ arbitrary, $c_4=c_5=1$, and $c_2=c_3=0$.
- The point symmetry $\mathrm{Y}_2$ corresponds to the equivalence transformation (4.4).
- The point symmetry

$$\mathrm{Y}_3=\frac{1}{r^2B^2}\frac{\partial}{\partial B^2}=\frac{1}{I}\frac{\partial}{\partial I}$$

  corresponds to the equivalence transformation (4.5).
- The point symmetry $\mathrm{Y}_4$ corresponds to the transformation (4.3) with $c_4$ arbitrary, $c_5=1$, and $c_1=c_2=c_3=0$.
- The point symmetry $\mathrm{Y}_5$ corresponds to the equivalence transformation with the infinitesimal generator

$$\mathrm{E}_2 = \psi\frac{\partial}{\partial \psi} + F\frac{\partial}{\partial F} + G\frac{\partial}{\partial G},$$

where $F = I(\psi)I'(\psi)$ and $G = P'(\psi)$. In particular, it corresponds to a linear combination of the generators arising from the equivalence transformation (4.3) with arbitrary parameters $c_4$ and $c_5$ together with $c_1 = c_2 = c_3 = 0$.

## V. RELATIONS BETWEEN INVARIANT SOLUTIONS

The PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14), its potential system $\mathbf{A\Psi}\{r,z;B^1,B^2,B^3,P,\psi\}$ (3.15), and the Bragg–Hawthorne equation $\mathbf{\Psi}\{r,z;\psi\}$ (3.16), are locally equivalent, i.e., their solution sets are the same, in the following sense.

- Every solution $(B^1(r,z),B^2(r,z),B^3(r,z),P(r,z),\psi(r,z))$ of the potential system (3.15) directly yields a solution $\psi(r,z)$ of (3.16), and a solution $(B^1(r,z),B^2(r,z),B^3(r,z),P(r,z))$ of the PDE system (3.14), by projection.
- Every solution $(B^1(r,z),B^2(r,z),B^3(r,z),P(r,z))$ of (3.14) yields a solution $\psi(r,z)$ of (3.16) satisfying

$$\psi_z = rB^1, \quad \psi_r = -rB^3$$

  for some $I(\psi) = rB^2$ and $P(\psi) = P(r,z)$. Such solution $\psi(r,z)$ is unique [modulo equivalence transformations (4.3) with $c_1 = c_2 = c_3 = 0$, $c_4 = c_5 = 1$]. Moreover, the quintuple $(B^1(r,z),B^2(r,z),B^3(r,z),P(r,z),\psi(r,z))$ is the corresponding solution of the potential system (3.15).
- For any prescribed pair of functions $(I(\psi),P(\psi))$, each solution $\psi = \psi(r,z)$ of the Bragg–Hawthorne equation (3.16) yields a unique solution $(B^1(r,z),B^2(r,z),B^3(r,z),P(r,z))$ (3.17) of the PDE system (3.14) and a corresponding unique solution $(B^1(r,z),B^2(r,z),B^3(r,z),P(r,z),\psi(r,z))$ of the potential system (3.15).

However, as it has been shown above, since the Bragg–Hawthorne equation (3.16) has two constitutive functions, while the PDE systems (3.14) and (3.15) have none, the symmetry classifications differ. In particular, for the Bragg–Hawthorne equation (3.16), one has a symmetry classification with respect to specific constitutive functions (Table I), whereas the PDE systems (3.14) and (3.15) have similar point symmetry structures given by the generators (4.1) and (4.2). It is now of interest to examine relations between families of invariant solutions arising from the different symmetry classifications.

In the Appendix, all invariant solutions arising from point symmetry reductions for the axially symmetric plasma equilibrium PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) are computed.

We now find invariant solutions arising from point symmetry reductions for the Bragg–Hawthorne equation $\mathbf{\Psi}\{r,z;\psi\}$ (3.16) and isolate new classes of solutions, i.e., solutions that *do not arise* as point symmetry-invariant solutions of the PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14).

### A. Reductions of the nonlinear Bragg–Hawthorne equation (3.16)

In the current section, we find solutions of the PDE (3.16) arising for each specific case listed in Table I. In particular, we obtain solutions invariant with respect to the admitted point symmetries in each nonlinear case listed in Table I and separable solutions in the linear case.

*Case 5.1.1: [$I(\psi)$, $P(\psi)$ arbitrary]* Solutions of the Bragg–Hawthorne equation (3.16) invariant with respect to $z$-translations can be constructed for any choice of the arbitrary functions $I(\psi)$, $P(\psi)$. In particular, here $\psi(r,z) = \Phi(r)$ satisfies the ordinary differential equation (ODE)

$$\Phi''(r) - \frac{1}{r}\Phi(r) + I(\Phi(r))I'(\Phi(r)) = -r^2P'(\Phi(r)). \tag{5.1}$$

For an arbitrary $\Phi(r)$, one can find $I(\Phi(r))$ and $P(\Phi(r))$ such that Eq. (5.1) is satisfied. However, as to be expected, the corresponding solutions of the plasma equilibrium PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) arise from its invariance under $z$-translations $\mathrm{Y}_1$ and are given by formula (A1b) in the Appendix.

*Case 5.1.2: [$I(\psi)I'(\psi)=\alpha_1 e^{\psi}$, $P'(\psi)=\alpha_2 e^{2\psi}$]* Here, it suffices to consider solutions invariant with respect to the symmetry $\mathrm{Z}_2$. [Considering solutions invariant with respect to a linear combination $\mathrm{Z}_1+\alpha\mathrm{Z}_2$, $\alpha\neq 0$, only adds an obvious $z$-translation.] $\mathrm{Z}_2$-invariant solutions are given by

$$\kappa = z/r, \quad \psi(r,z) = 2\log r + \Phi(\kappa),$$

which reduces the Bragg–Hawthorne equation (3.16) to the ODE

$$(1+\kappa^2)\Phi''(\kappa) + 3\kappa\Phi'(\kappa) + 4 + \alpha_1 e^{\Phi(\kappa)} + \alpha_2 e^{2\Phi(\kappa)} = 0. \tag{5.2}$$

The authors are unaware of any closed-form exact solutions of the nonlinear ODE (5.2).

For each solution $\Phi(\kappa)$ of the ODE (5.2), the corresponding solution of the axially symmetric MHD equilibrium system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) is given by (3.13) with

$$B^1(r,z) = \frac{1}{r}\frac{\partial}{\partial z}\Phi\left(\frac{z}{r}\right), \quad B^2(r,z) = \frac{1}{r}\sqrt{2\alpha_1 e^{\Phi(z/r)+\gamma}}, \quad B^3(r,z) = -\frac{1}{r}\frac{\partial}{\partial r}\Phi\left(\frac{z}{r}\right),$$

$$P = P_0 + \tfrac{1}{2}\alpha_2 e^{2\Phi(z/r)},$$

$$\alpha_1, \alpha_2, \gamma, P_0 = \text{const}. \tag{5.3}$$

By comparison with the solutions presented in the Appendix one can observe that the solutions (5.3) do not arise as invariant solutions with respect to any point symmetry of the MHD equilibrium system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14).

*Case 5.1.3: [$I(\psi)I'(\psi)=\psi^{1+1/\gamma}$, $P'(\psi)=\psi^{1+2/\gamma}$]* In this case, one can demonstrate that the corresponding invariant solutions are obtainable from a reduction of the MHD equilibrium system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) with respect to its symmetry $\mathrm{Y}_4+\alpha\mathrm{Y}_2+\beta\mathrm{Y}_3+\delta\mathrm{Y}_5$, and the corresponding exact solution is given by formula (A18) in the Appendix.

*Case 5.1.4: [$I(\psi)I'(\psi)=\psi^{-3}$, $P'(\psi)=\psi^{-7}$]* In this particular power nonlinearity case ($\gamma=-1/4$), the Bragg–Hawthorne equation (3.16) has an additional symmetry $\mathrm{Z}_4$. Consider invariant solutions with respect to the most general symmetry

$$\mathrm{Z} = \alpha\mathrm{Z}_1 + \beta\mathrm{Z}_3 + \delta\mathrm{Z}_4, \quad \alpha,\beta,\delta = \text{const}, \quad \delta\neq 0.$$

Such solutions have the form

$$w = w(r,z) = \frac{2(\alpha+\beta z) + \delta z^2 + r^2}{r}, \quad \psi(r,z) = r^{1/2}\Phi(w). \tag{5.4}$$

With respect to the *Ansatz* (5.4), the Bragg–Hawthorne equation (3.16) reduces to the nonlinear ODE

$$(4\beta^2 - 8\alpha\delta + w^2)\Phi''(w) + 2w\Phi'(w) - \tfrac{3}{4}\Phi(w) + (\Phi(w))^{-3} + (\Phi(w))^{-7} = 0. \tag{5.5}$$

The ODE (5.5), in general, does not belong to any ODE class for which exact closed-form solutions are known. For the particular case $4\beta^2-8\alpha\delta=0$, ODE (5.5) is invariant under scalings in $w$ and, accordingly, can be reduced to the first-order ODE

$$4X^7(Y'(X)-Y^2(X))+(3X^8-4(X^4+1))Y^3(X)=0,$$

where $X=\Phi(w)$ and $Y(X)=1/(w\Phi'(w))$.

For each solution $\Phi(w)$ of the ODE (5.5), the corresponding solution of the axially symmetric MHD equilibrium system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) is given by (3.13) with

$$B^1(r,z)=\frac{1}{r}\frac{\partial}{\partial z}\Phi(w(r,z)),\quad B^2(r,z)=\frac{1}{r}\sqrt{\gamma-\Phi^{-2}(w(r,z))},$$

$$B^3(r,z)=-\frac{1}{r}\frac{\partial}{\partial r}\Phi(w(r,z)),\quad P=P_0-\frac{1}{6}\Phi^{-6}(w(r,z)),$$

$$\gamma,P_0=\text{const.}\tag{5.6}$$

The solutions (5.6) do not arise as invariant solutions with respect to any point symmetry of the MHD equilibrium system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) (cf. Appendix).

### B. Exact solutions of the linear Bragg-Hawthorne equation (3.16)

We now consider case 5 in Table I, i.e., the linear Bragg–Hawthorne equation

$$\psi_{rr}-\frac{1}{r}\psi_r+\psi_{zz}+a_1\psi+a_2+r^2(a_3\psi+a_4)=0,\quad a_1,\ldots,a_4=\text{const},\tag{5.7}$$

with $I(\psi)I'(\psi)=a_1\psi+a_2$ and $P'(\psi)=a_3\psi+a_4$. From its linearity, the PDE (5.7) has an infinite number of symmetries. Instead of looking for symmetry-invariant solutions, we seek separable solutions of (5.7).

By comparison with the Appendix, it will be seen that none of the exact solutions obtained in the present section arises as an invariant solution with respect to any point symmetry of the MHD equilibrium system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14).

First we note that through either using equivalence transformations (4.3) or, directly, the substitution

$$\psi(r,z)=\Psi(r,z)-\tfrac{1}{2}(a_4r^2+a_2)z^2,$$

one can convert PDE (5.7) into the linear homogeneous PDE

$$\Psi_{rr}-\frac{1}{r}\Psi_r+\Psi_{zz}+a_1\Psi+a_3r^2\Psi=0.\tag{5.8}$$

Seeking separable solutions $\Psi(r,z)=R(r)Z(z)$ of (5.8), one obtains

$$Z''(z)=\lambda Z(z),\tag{5.9a}$$

$$R''(r)-\frac{1}{r}R'(r)+((a_1+\lambda)+a_3r^2)R(r)=0,\quad \lambda=\text{const.}\tag{5.9b}$$

Consequently, one has the following cases.

*Case 5.2.1: ($a_3\neq 0$)* Following Ref. 21, assume that $a_1=\alpha^2$ and $a_3=-4\beta^2$. The substitution $x=2\beta r^2$ converts the ODE (5.9b) into

$$xR''(x)+\frac{\alpha^2+\lambda-2\beta x}{8\beta}R(x)=0,\tag{5.10}$$

which is a classical Whittaker's differential equation with $\mu=(\alpha^2+\lambda)/8\beta$, $\nu=1/2$,[33] and thus has a general solution in terms of the two Whittaker functions,

$$R(x) = C_1 M_{\mu,\nu}(x) + C_2 W_{\mu,\nu}(x). \tag{5.11}$$

Solutions from the family (5.11) with specific relations between physical plasma parameters $\alpha$ and $\beta$ can have a significantly simpler form and a transparent physical meaning, which is demonstrated as follows. In (5.10), the substitution $R(x)=e^{-x/2}U(x)$, yields the ODE

$$xU''(x) - xU'(x) + \frac{\alpha^2+\lambda}{8\beta}U(x) = 0. \tag{5.12}$$

In the important special case where $n=(\alpha^2+\lambda)/8\beta$ is a non-negative integer, there exist polynomial solutions of (5.12) related to the Laguerre polynomials,[21]

$$U(x) = L_n^*(x) = -\frac{x}{n!}e^x\frac{d^n}{dx^n}(e^{-x}x^{n-1}), \quad L_0^*(x) = -1. \tag{5.13}$$

*Case 5.2.1.1: (z-periodic or quasiperiodic solutions)* When $\lambda=-\omega^2<0$, ODE (5.9a) has the obvious periodic solutions given by $Z(z)=A\sin(\omega z)+B\cos(\omega z)$. It follows that in order for the radial part $U(x)$ to have a polynomial form, one must require

$$\omega = \omega_n = \sqrt{\alpha^2 - 8\beta n}, \quad n=0,\ldots,N, \quad N = \left[\frac{\alpha^2}{8\beta}\right].$$

Then the corresponding solutions of the Bragg–Hawthorne equation (5.8) are given by an arbitrary linear combination involving $2N+2$ arbitrary constants,

$$\Psi(r,z) = e^{-\beta r^2}\sum_{n=0}^{N} L_n^*(2\beta r^2)(a_n\cos(\omega_n z) + b_n\sin(\omega_n z)). \tag{5.14}$$

Solution (5.14) represents $z$-periodic or quasi-$z$-periodic flux functions of global axially symmetric plasma equilibria, satisfying important physical conditions. In particular, (a) the corresponding plasma magnetic field and pressure (3.17), as well as electric current density $\mathbf{J}=\mathrm{curl}\,\mathbf{B}$, are bounded functions in $\mathbb{R}^3$, and (b) in the limit $r\to\infty$, one has $\mathbf{B}\to 0$, $\mathbf{J}\to 0$, $P\to\mathrm{const}$. For further details, see Ref. 21.

*Case 5.2.1.2: (Axially symmetric plasma equilibria in half-space $z>0$)* When $\lambda=\omega^2>0$, ODE (5.9a) has the general solution $Z(z)=A\exp(\omega z)+B\exp(-\omega z)$, $\omega>0$. For the half-space $z>0$, one must take $Z(z)=\exp(-\omega z)$. In order for the radial part $U(x)$ to have a polynomial form, one again requires

$$\omega = \omega_n = \sqrt{8\beta n - \alpha^2}, \quad n=N,N+1,\ldots, \quad N \geq \left[\frac{\alpha^2}{8\beta}\right].$$

The corresponding solutions of the Bragg–Hawthorne equation (5.8) are given by a general linear combination

$$\Psi(r,z) = e^{-\beta r^2}\sum_{n=N}^{N+m} a_n L_n^*(2\beta r^2)\exp(\omega_n z), \tag{5.15}$$

which contains an arbitrary number of terms $m\geq 0$, each involving a free constant $a_n$. Plasma equilibrium configuration (3.17) corresponding to flux function (5.15) have finite total magnetic energy $E=\int\int\int(\mathbf{B}^2/2)dV$ in the half-space $z>0$.[21] [For Euler equations (3.2), this corresponds to a finite total kinetic energy.]

*Case 5.2.2: ($a_3=0$)* In this case, ODE (5.9b) is related to Bessel's equation of order one and the general solution is given by

$$R(r) = r(C_1 J_1(r\sqrt{a_1+\lambda}) + C_2 Y_1(r\sqrt{a_1+\lambda})), \quad C_1, C_2 = \text{const}.$$

Since $(d/dr)(rJ_1(\alpha r)) = \alpha J_0(\alpha r)$, it follows that for any nontrivial choice of constants $C_1, C_2, a_1, \lambda$, the corresponding solution $(\mathbf{B}(r,z), P(r,z))$ (3.17) of the MHD equilibrium system (3.14) is singular on the symmetry axis $r=0$.

*Case 5.2.3: ($a_1 = p = 0$)* If $a_3 > 0$, let $a_3 = 4\gamma^2$. Then the general solution of the ODE (5.9b) is given by

$$R(r) = C_1 \sin(\gamma r^2) + C_2 \cos(\gamma r^2), \quad C_1, C_2 = \text{const},$$

and if $a_3 < 0$, letting $a_3 = -4\beta^2$, one obtains

$$R(r) = C_1 \sinh(\beta r^2) + C_2 \cosh(\beta r^2).$$

The corresponding solution for the $z$-component is given by

$$Z(z) = \mu z + \nu, \quad \mu, \nu = \text{const}.$$

The corresponding solution $(\mathbf{B}(r,z), P(r,z))$ (3.17) of the axially symmetric MHD equilibrium system (3.14) can be regular and bounded in $r$ in each cross-sectional plane $z$=const. For example, when $a_2 = a_4 = 0$, $I(\psi) = 0$, $P(\psi) = P_0 + a_3\psi^2/2$, $C_1 = 1$, and $C_2 = 0$, one has

$$\mathbf{B}(r,z) = \frac{\mu}{r}\sin\left(\frac{\sqrt{a_3}}{2}r^2\right)\mathbf{e}_r - \sqrt{a_3}(\mu z + \nu)\cos\left(\frac{\sqrt{a_3}}{2}r^2\right)\mathbf{e}_z,$$

$$P(r,z) = P_0 + \frac{a_3}{2}\sin^2\left(\frac{\sqrt{a_3}}{2}r^2\right)\psi^2/2. \tag{5.16}$$

*Case 5.2.4: ($a_1 = a_3 = \lambda = 0$)* This case corresponds to trivial solutions

$$\Psi(r,z) = (C_1 + C_2 r^2)(\mu z + \nu), \quad C_1, C_2, \mu, \nu = \text{const}$$

of the PDE (5.8), which yield unbounded solutions of the MHD equilibrium PDE system (3.14).

## VI. DISCUSSION

Usually a conservation law of a given PDE system yields a potential variable that is nonlocal, i.e., it is not expressible in terms of the independent variables as well as the dependent variables and their derivatives of the given PDE system. However, in this paper, we have presented examples where a local conservation law of a given PDE system with two independent variables yields a potential variable, which is also a local variable of the given PDE system and, in particular, a function of its dependent and independent variables. Lemma 1 gives a simple necessary and sufficient condition for such a situation.

As physical examples, we considered two-dimensional reductions of the PDE system of incompressible equilibrium Euler equations of fluid dynamics (or, equivalently, static MHD equilibrium equations) with respect to helical, axial, and translational symmetries. It was shown that each such reduction (Sec. III) has a conservation law which yields a *local* potential variable. In terms of its corresponding local potential variable, each PDE system further reduces to a scalar equation in terms of the potential variable: the JFKO equation (3.12), the Bragg–Hawthorne equation (3.16), and the "flat Bragg-Hawthorne equation" (3.20), respectively. It follows that these three well-known nonlinear second-order PDEs are locally related to the original two-dimensional PDE systems from which they are derived.

Due to the local relations between each such two-dimensional PDE system and its potential equation, one might expect a straightforward relation between correspondence local symmetries. As an example, we studied the point symmetry classifications of the Bragg–Hawthorne (potential) equation (3.16) and the axially symmetric version (3.14) of the system of MHD equilibrium equations (3.3). It was shown that local symmetry relations between these systems are restricted due to the fact that the dependent variables $B^2$ and $P$ of system (3.14) relate to *arbitrary constitutive functions* in the potential (Bragg–Hawthorne) equation (3.16). Moreover, the introduced potential variable has a particular relationship with the dependent variables $B^1$ and $B^3$ of system (3.14). As a consequence of this relationship, most point symmetries of the PDE system (3.14) correspond to equivalence transformations of the potential equation (3.16). Conversely, further local symmetries arising in special cases of the symmetry classification of the potential equation (3.16) are restricted symmetries holding only for special classes (i.e., subsets) of solutions of the initial PDE system (3.14). Importantly, in the case when $P(\psi)$ is a quadratic function and $I(\psi)$ is a linear function (up to equivalence transformations), the Bragg–Hawthorne equation (3.16) becomes *linear*, whereas the PDE system (3.14) is not explicitly linear.

Finally, it was of interest to classify and compare solutions arising from point symmetry reductions of the axially symmetric MHD equilibrium system (3.14) with those arising from point symmetry reductions of the Bragg–Hawthorne equation (3.16). It was shown that all symmetry-invariant solutions of the PDE system (3.14) arose as symmetry-invariant solutions of the Bragg–Hawthorne equation (3.16). However, the converse was not true. In particular, it was shown that there exist symmetry-invariant solutions of the Bragg–Hawthorne equation (3.16) that do not arise as symmetry-invariant solutions of the axially symmetric MHD equilibrium system (3.14). These include a wide class of solutions with interesting physical behavior[21] that arise for the linear Bragg–Hawthorne equation.

The principal results of this paper can be summarized as follows.

(1) Local potential systems can arise in practical situations.
(2) Local potential variables can be useful for the computation of further solutions of a given PDE system. In particular, point symmetries of such local potential systems can correspond to restricted symmetries of a given PDE system in the sense that only a submanifold of solutions of the given PDE system is invariant.

There exist other approaches for obtaining solutions of PDEs from symmetry-related *Ansätze*. These include the nonclassical method[34,35] where one seeks solutions of a given PDE system that arise from restricted ("nonclassical") symmetries that leave invariant a PDE system that includes the given PDE system and the invariant surface condition satisfied by the invariant solution. Here, the invariant submanifold is the invariant solution itself. For many PDEs, this method has been fruitful in obtaining solutions that do not arise as symmetry-invariant solutions.

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## APPENDIX: INVARIANT SOLUTIONS OF THE AXIALLY SYMMETRIC PLASMA EQUILIBRIUM SYSTEM $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14) WITH RESPECT TO ITS POINT SYMMETRIES

Due to the local relationship between the PDE system (3.14) and its potential system (3.15), it follows that all local symmetries of the potential system (3.15) are local symmetries of the PDE system (3.14). At the same time, in terms of finding solutions, it is preferable to use the original PDE system to obtain larger families of solutions if one seeks solutions through an extension based on the invariant solution *Ansatz*.[36]

As is well known, to obtain symmetry-invariant solutions of a PDE system, it is essential that the symmetry generator have nonzero components corresponding to independent variables. Therefore, in seeking symmetry-invariant solutions of the PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14), one should only consider linear combinations of symmetries (4.2) involving generator $\mathrm{Y}_1$ or $\mathrm{Y}_4$. (It is not necessary to consider linear combinations $a\mathrm{Y}_1+b\mathrm{Y}_4$, $ab\neq 0$, since they differ from invariant solutions with respect to $\mathrm{Y}_4$ only by a $z$-translation.)

We now consider nontrivial combinations of symmetry generator $\mathrm{Y}_1$ or $\mathrm{Y}_4$ with other generators in (4.2) and study whether the corresponding invariant solutions can be obtained as invariant solutions of the Bragg-Hawthorne equation $\mathbf{\Psi}\{r,z;\psi\}$ (3.16) with respect to its point symmetries listed in Table I.

### 1. Solutions invariant with respect to $\mathrm{Y}_1$

Here, the symmetry invariants are $r$, $B^1$, $B^2$, $B^3$, and $P$, and the invariant solution is sought in the form $B^i=B^i(r)$, $i=1,2,3$, $P=P(r)$. Consequently, this yields two families of solutions

$$B^1(r)=\frac{B_0^1}{r},\quad B^2(r)=\frac{B_0^2}{r},\quad B^3(r)=B_0^3,\quad P(r)=P_0, \tag{A1a}$$

$$B^1(r)=0,\quad B^2=A(r),\quad B^3=C(r),\quad P=P_0-\frac{1}{2}(A^2(r)+C^2(r))+\int\frac{A^2(r)}{r}dr, \tag{A1b}$$

where $B_0^i,P_0=\text{const}$, and $A(r)$ and $C(r)$ are arbitrary functions. We now check whether the invariant solutions (A1) arises as invariant solutions of the Bragg–Hawthorne equation (3.16).

For the solution family (A1a), from (3.17), one identifies $\psi_z=B_0^1$, $\psi_r=-B_0^3 r$, $I(\psi)=B_0^2$, and $P(\psi)=P_0$. Hence, one has

$$\psi(r,z)=-\tfrac{1}{2}r^2B_0^3+B_0^1 z+\text{const},\quad I(\psi)I'(\psi)=P(\psi)=0,$$

which corresponds to a basic solution of the PDE $g_{rr}-(1/r)g_r+g_{zz}=0$ arising in case 5d in Table I, i.e., an invariant solution of the Bragg–Hawthorne equation (3.16) with respect to the symmetry $\mathrm{Z}_\infty$.

For the solution family (A1b), one has $\psi_z=0$. Instead of $A(r)$ and $C(r)$, in (A1b) one may equivalently treat $P=P(r)$ and $I=I(r)$ as arbitrary functions. Then one finds

$$\psi(r)=C_0+C_1r^2+\int\left[\int^r\left(\frac{I'(s)}{s}+sP'(s)\right)ds\right]dr,$$

which arises as an invariant solution of the Bragg–Hawthorne equation (3.16) [with arbitrary $P(\psi)=P(r)$ and $I(\psi)=I(r)$] with respect to $\mathrm{Z}_1=\partial/\partial z$.

### 2. Solutions invariant with respect to $\mathrm{Y}_1+\alpha\mathrm{Y}_2+\beta\mathrm{Y}_3$, $\alpha^2+\beta^2>0$

Here, the similarity variable is also the radius $r$, and one obtains

$$B^1(r,z)=\frac{1}{C_1r},\quad B^2(r,z)=\frac{1}{r}\sqrt{2\beta z+C_2-\frac{1}{4}\alpha\beta C_1^2r^4+r^2\left(\beta^2C_1^2\left(\frac{1}{2}-\log r\right)-\beta C_1C_3\right)},$$

$$B^3(r,z)=\tfrac{1}{2}\alpha C_1r^2+\beta C_1\log r+C_3,$$

$$P(r,z)=\alpha z-\tfrac{1}{8}(\alpha^2C_1^2r^4+4\alpha C_1C_3r^2)+\tfrac{1}{4}\alpha\beta C_1^2r^2(1-2\log r)+C_4,$$

$$C_1,\ldots,C_4=\text{const}. \tag{A2}$$

Using (3.17), it is straightforward to show that the solution (A2) corresponds to the solution

$$\psi(r,z) = \frac{z}{C_1} - \frac{1}{8}\alpha C_1 r^4 - \frac{1}{2}C_3 r^2 + C_5 + \frac{1}{4}\beta C_1 r^2(1 - 2\log r), \quad C_5 = \text{const} \tag{A3}$$

of the Bragg–Hawthorne equation (3.16). From (3.17), one identifies $I(\psi)=\sqrt{2\beta C_1\psi+\text{const}}$, $P(\psi)=\alpha C_1\psi$. This corresponds to case 5 in Table I, i.e., a linear Bragg–Hawthorne equation (3.16) with $I(\psi)I'(\psi)=2\beta$, $P'(\psi)=\alpha C_1$. It is easy to see that the solution (A3) can be represented as $\psi(r,z)=\psi_{\text{eq}}(r,z)+\psi_{\text{hom}}(r,z)$, where

$$\psi_{\text{hom}}(r,z) = \frac{z}{C_1} - \frac{1}{2}C_3 r^2 + C_5$$

is an obvious solution of the homogeneous Bragg–Hawthorne equation, and the part

$$\psi_{\text{eq}}(r,z) = -\tfrac{1}{8}\alpha C_1 r^4 - \tfrac{1}{4}\beta C_1 r^2(1 - 2\log r),$$

arises from the equivalence transformation (4.3) of the homogeneous Bragg–Hawthorne equation into one with $I(\psi)I'(\psi)=2\beta$, $P'(\psi)=\alpha C_1$.

### 3. Solutions invariant with respect to $\mathbf{Y}_1+\alpha\mathbf{Y}_2+\beta\mathbf{Y}_3+\delta\mathbf{Y}_5$, $\delta\neq 0$

Here, the similarity variable is again the cylindrical radius $r$, and one readily finds the following form of the invariant solutions

$$B^1(r,z) = Q^1(r)e^{\delta z}, \quad B^2(r,z) = \frac{1}{r}\sqrt{Q^2(r)e^{2\delta z} - \frac{\beta}{\delta}},$$

$$B^3(r,z) = Q^3(r)e^{\delta z}, \quad P(r,z) = P_0 + Q^4(r)e^{2\delta z}, \quad P_0 = \text{const}. \tag{A4}$$

After substitution of the *Ansatz* (A4) into the PDE system $\mathbf{A}\{r,z;B^1,B^2,B^3,P\}$ (3.14), one obtains a system of four first-order ODEs in terms of the four unknown functions $Q^i(r)$, which, in turn, can be reduced to a fourth-order nonlinear ODE for $Q^1(r)$. Furthermore, using two obvious first integrals and the substitution $x=r^2$, $A(x)=Q^1(r)$, one can show that this fourth-order ODE can be reduced to the second-order linear ODE

$$(4x^2 - 1)A''(x) + 4xA'(x) + (4a_2x^2 + 4a_1x - 1)A(x) = 0, \quad a_1,a_2 = \text{const}. \tag{A5}$$

The remaining part of solution (A4) is given by

$$Q^3(r) = -\frac{1}{\delta r}(r(Q^1)'(r) + Q^1(r)),$$

$$Q^2(r) = (4a_1 - \delta^2)\Phi^2(r), \quad Q^4(r) = 2a_2\Phi^2(r),$$

$$\Phi(r) = \frac{r}{\delta}Q^1(r). \tag{A6}$$

Correspondingly, the flux function is given by

$$\psi(r,z) = \Phi(r)e^{\delta z},$$

and hence for such invariant solutions, one has the relations

$$P(r,z) = P_0 + 2a_2(\psi(r,z))^2, \quad I(r,z) \equiv rB^2(r,z) = \sqrt{(4a_1 - \delta^2)(\psi(r,z))^2 - \frac{\beta}{\delta}},$$

which corresponds to the linear dependence $P'(\psi)=4a_2\psi$ and $I(\psi)I'(\psi)=(4a_1-\delta^2)\psi$. In Sec. V B, it is seen that the invariant solutions (A4) are a subset of the solution set of the linear Bragg–Hawthorne equation (Table I, case 5) obtained by separation of variables. In particular, for specific relations between constants $a_1, a_2$, solution (A4) is expressible in terms of elementary (exponential, Gaussian, and polynomial) functions.

To solve the ODE (A5), we let $S(x)=A(x)\sqrt{x}$ and obtain the ODE

$$xS''(x) + (a_2 x + a_1)S(x) = 0. \tag{A7}$$

(1) When $a_1, a_2 \neq 0$, Eq. (A7) is related to Whittaker's linear ODE, and thus one obtains its general solution

$$Q^1(r) = r^{-1}[C_1 M_{-(1/2)\mathrm{i}a_1 a_2,1/2}(2\mathrm{i}a_2^{-1}r^2) + C_2 W_{-(1/2)\mathrm{i}a_1 a_2,1/2}(2\mathrm{i}a_2^{-1}r^2)], \tag{A8}$$

where $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ are Whittaker functions.

(2) When $a_2=0$, Eq. (A7) can be transformed into Bessel's equation, and one has

$$Q^1(r) = C_1 J_1(2\sqrt{a_1}r) + C_2 Y_1(2\sqrt{a_1}r), \quad C_1, C_2 = \text{const}, \tag{A9}$$

where $J_1(z)$ and $Y_1(z)$ are Bessel functions of order one.

(3) When $a_1=0$, Eq. (A7) has the general solution

$$Q^1(r) = \frac{1}{r}C_1 \sin(\sqrt{a_2}r^2) + C_2 \cos(\sqrt{a_2}r^2), \quad C_1, C_2 = \text{const}. \tag{A10}$$

(4) When $a_1=a_2=0$, the general solution of ODE (A7) is given by

$$Q^1(r) = C_1 r + \frac{C_2}{r}, \quad C_1, C_2 = \text{const}. \tag{A11}$$

Note that solutions (A10) and (A11) correspond to constant pressure: $Q(r)=0$, $P(\psi)=\text{const}$.

#### 4. Solutions invariant with respect to $Y_4$

The similarity variable following from the scaling symmetry $Y_4$ is $\kappa=z/r$, and one readily finds the invariant solution

$$B^1(\kappa) = -C_1\kappa + C_2\sqrt{1+\kappa^2},$$

$$B^2(\kappa) = \pm\sqrt{C_1[C_2 \operatorname{arcsinh}\kappa - C_1\kappa^2 + C_2\kappa\sqrt{1+\kappa^2} - C_3]},$$

$$B^3(\kappa) = -C_2 \operatorname{arcsinh}\kappa + C_3, \quad P(\kappa) = C_4, \quad C_1, \ldots, C_4 = \text{const}. \tag{A12}$$

It is straightforward to show that solution (A12) corresponds to the solution

$$\psi(r,z) = \frac{1}{2}\left[C_2\left(r^2 \operatorname{arcsinh}\frac{z}{r} + z\sqrt{z^2+r^2}\right) - C_1 z^2 - C_3 r^2\right] \tag{A13}$$

of the Bragg–Hawthorne equation (3.16) with $I(\psi)=\sqrt{2C_1\psi}$, $P(\psi)=C_4=\text{const}$. This corresponds to case 5c in Table I: $P'(\psi)=0$, $I'(\psi)I(\psi)=\text{const}$ and, indeed, solution (A13) is an invariant solution of the Bragg–Hawthorne equation with respect to its point symmetry $Z_7$.

### 5. Solutions invariant with respect to $Y_4+\alpha Y_2+\beta Y_3$, $\alpha^2+\beta^2>0$

Again, equivalently, one can consider the symmetry $X_4+\alpha X_5$ of the potential system (3.15). Since $B^2(r,z)=I(\Psi)/r$, $P(r,z)=P(\Psi)$, the form of the corresponding invariant solutions is given by

$$\kappa = z/r, \quad \psi(r,z) = r^2\Phi(\kappa),$$

$$B^i(r,z) = Q^i(\kappa), \quad i=1,3, \quad B^2(r,z) = \frac{1}{r}\sqrt{R(\kappa)r^2-\beta}, \quad A = \text{const},$$

$$P(r,z) = \alpha \log r + Q(\kappa). \tag{A14}$$

After substitution of the *Ansatz* (A14) into the PDE system (3.15), one finds that $\Phi(\kappa)$ satisfies the nonlinear second-order ODE

$$(\kappa^2+1)\Phi''(\kappa) - \kappa\Phi'(\kappa) + \frac{1}{2}\left(\frac{\alpha}{\Phi(\kappa)}+K\right) = 0, \quad K = \text{const}. \tag{A15}$$

Each solution $\Phi(\kappa)$ of the ODE (A15) yields a corresponding invariant solution (A14) of the axially symmetric PDE system (3.14) through the formulas

$$Q^1(r,z) = \Phi'(z/r), \quad Q^2(r,z) = \frac{1}{r}\sqrt{K\Phi(z/r)-\beta}, \quad Q^3(r,z) = \kappa\Phi'(z/r) - 2\Phi(z/r),$$

$$P(r,z) = P_0 + \frac{\alpha}{2}\log(r^2\Phi(z/r)). \tag{A16}$$

It follows that in this case, $I(\psi)I'(\psi)=K/2$, $P'(\psi)=\alpha\psi^{-1}/2$, which corresponds to case 3 in Table I ($\gamma=-1$, $\alpha_1=A/2$, $\alpha_2=\alpha/2$). Indeed, as shown in Sec. V A, solution (A16) corresponds to an invariant solution of the Bragg–Hawthorne equation (3.16) with respect to the scaling symmetry $Z_3$ with $\gamma=-1$, and the ODE (A15) is the respective reduction of the Bragg–Hawthorne equation (3.16).

### 6. Solutions invariant with respect to $Y_4+\alpha Y_2+\beta Y_3+\delta Y_5$, $\delta\neq 0$

Here, one may again equivalently consider invariant solutions with respect to the symmetry $X_4+\alpha X_2+\beta X_3+\delta X_5$ of the potential system (3.15). Solving the corresponding characteristic ODEs, one finds the invariant solution form

$$\kappa = z/r, \quad \psi(r,z) = r^{2+\delta}\Phi(\kappa),$$

$$B^i(r,z) = r^\delta Q^i(\kappa), \quad i=1,3, \quad B^2(r,z) = \frac{1}{r}\sqrt{K(\kappa)r^{2\delta+2} - \frac{\beta}{\delta+1}},$$

$$P(r,z) = Q(\kappa)r^{2\delta} - \frac{\alpha}{2\delta}. \tag{A17}$$

After substitution of the *Ansatz* (A17) into the PDE system (3.15), one finds

$$Q^i(\kappa) = \Phi'(\kappa), \quad Q^3(\kappa) = \kappa\Phi'(\kappa) - (\delta+2)\Phi(\kappa),$$

$$B^2(r,z) = \frac{1}{r}\sqrt{K(\Phi(\kappa))^{(2\delta+2)/(\delta+2)} - \frac{\beta}{\delta+1}},$$

$$P(r,z) = P_0 + Q(\Phi(\kappa))^{(2\delta)/(\delta+2)}, \quad P_0, K, Q = \text{const}, \tag{A18}$$

where $\Phi(\kappa)$ satisfies the ODE

$$(1+\kappa^2)\Phi''(\kappa) - (2\delta+1)\kappa\Phi'(\kappa) + \delta(\delta+2)\Phi(\kappa) + \tilde{R}(\Phi(\kappa))^{1-2/(\delta+2)} + \tilde{Q}(\Phi(\kappa))^{1-4/(\delta+2)} = 0,$$

$$\tilde{R}, \tilde{Q} = \text{const}. \tag{A19}$$

Using $B^2(r,z)=I(\psi)/r$, $P(r,z)=P(\psi)$ in (A17), one obtains $I(\psi)=\sqrt{K\psi^{(2\delta+2)/(\delta+2)}-\beta/(\delta+1)}$, $P(\psi)=Q\psi^{2\delta/(\delta+2)}-\alpha/2\delta$, $K,Q=$const, and hence

$$I(\psi)I'(\psi) = \tilde{K}\psi^{1-2/(\delta+2)}, \quad P'(\psi) = \tilde{Q}\psi^{1-4/(\delta+2)}, \quad \tilde{K}, \tilde{Q} = \text{const}.$$

Thus, the invariant solution (A17) corresponds to invariant solution of the Bragg–Hawthorne equation (3.16) with respect to the scaling symmetry $Z_3$ [Table I, case 3, $\gamma=-(\delta+2)/2$]. In particular, the projection of the symmetry $X_4+\alpha X_2+\beta X_3+\delta X_5$ on the space of variables $(r,z,\psi)$ of the Bragg–Hawthorne equation directly yields symmetry $Z_3$. Moreover, as shown in Sec. V A, Eq. (A19) is the corresponding reduction of the Bragg–Hawthorne equation (3.16).


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