The General Similarity Solution of the Heat Equation

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1. Introduction. A systematic approach is given for finding similarity solutions to partial differential equations and, in particular, the heat equation, by the use of transformation groups. New solutions to the heat equation are obtained.

The application of group theory to the solution of partial differential equations was first considered by Lie [1] and later by Ovsjannikov [2] and Matschat and Müller [3]. If a one-parameter group of transformations leaves invariant an equation and its accompanying boundary conditions, then the number of variables can be reduced by one. The functional form of the solution can be deduced by solving a first order partial differential equation derived from the infinitesimal version of the global group. The functional form of the solution for two independent variables (x, t), say, is

$$u(x, t) = F(x, t, \eta, f(\eta)) \tag{1}$$

where η is called the similarity variable. The dependence of F on x, t, $\eta(x, t)$ (for calculation purposes, as will be seen later, it is convenient to isolate η), and $f(\eta)$ is known explicitly from invariance considerations. $f(\eta)$ satisfies some ordinary differential equation obtained by substituting the form (1) into the given partial differential equation. The form (1) is called the general similarity solution. Initially no special boundary conditions are imposed since eventually we use the invariants of the group to establish the boundary conditions.

2. Formulation of invariance. Let

$$u = \theta(x, t) \tag{2}$$

be a solution of a certain partial differential equation

$$\mathfrak{M}[u] = 0 \tag{3}$$

defined over a region R in the (x, t) plane (Fig. 1) on which boundary conditions

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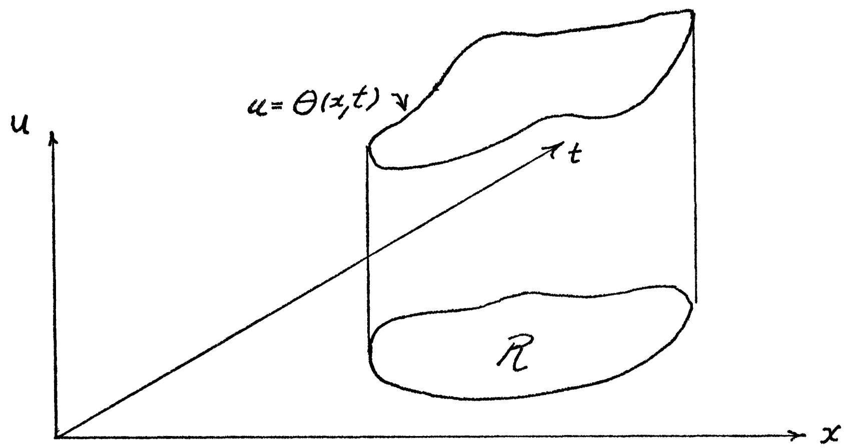

FIG. 1

$B_\alpha(u, x, t) = 0$ are prescribed on curves $\omega_\alpha(x, t) = 0$. We have in mind a unique solution defining a single surface $u = \theta(x, t)$ in the (u, x, t) space (Fig. 1). We consider a group of transformations, depending on the parameter ϵ, which takes the (u, x, t) space into itself:

$$\left.\begin{cases} x' = x'(x, t, u; \epsilon) \\ t' = t'(x, t, u; \epsilon) \\ u' = u'(x, t, u; \epsilon) \end{cases}\right\}. \tag{4}$$

Under the transformation (4), in general:

(5) (i) region $R \rightarrow$ region R'

(6) (ii) solution surface $u = \theta(x, t) \rightarrow u' = \theta'(x', t')$.

Invariance is defined as follows:

(7) (i) equation (3) is left invariant, *i.e.*, $\mathfrak{M}'[u'] = 0$ iff $\mathfrak{M}[u] = 0$ where $\mathfrak{M}'[u']$ is obtained from $\mathfrak{M}[u]$ if (x, t, u) is replaced by (x', t', u')

(8) (ii) the boundary conditions and boundary curves are left invariant, *i.e.*, region R' = region R, $B_\alpha(u', x', t') = 0$ on $\omega_\alpha(x', t') = 0$ for each α.

Then, assuming a unique solution to (3) over R with the associated boundary conditions, the solution surface must go into itself, *i.e.*, $u'(x, t, \theta(x, t); \epsilon) = \theta(x', t')$, and hence the functional form of θ can be deduced.

Now we consider the infinitesimal transformations corresponding to (4).

$$\left.\begin{cases} x' = x + \epsilon X(x, t, u) + O(\epsilon^2) \\ t' = t + \epsilon T(x, t, u) + O(\epsilon^2) \\ u' = u + \epsilon U(x, t, u) + O(\epsilon^2) \end{cases}\right\}. \tag{9}$$

The infinitesimal version of the invariance condition (8) is thus derived from

$$\theta(x + \epsilon X, t + \epsilon T) = \theta(x, t) + \epsilon U(x, t, \theta) + O(\epsilon^2).$$

Expanding the latter expression and equating $O(\epsilon)$ terms, we have

$$X(x, t, \theta) \frac{\partial \theta}{\partial x} + T(x, t, \theta) \frac{\partial \theta}{\partial t} = U(x, t, \theta). \tag{10}$$

(10) is the general partial differential equation of an invariant surface.

The characteristic equations corresponding to (10) are

$$\frac{dx}{X(x, t, \theta)} = \frac{dt}{T(x, t, \theta)} = \frac{d\theta}{U(x, t, \theta)}. \tag{11}$$

In principle (11) is solvable, and if X/T is independent of θ, we obtain the similarity form (1) for the solution θ. The similarity variable

$$\eta(x, t) = \text{const.} \tag{12}$$

is the integral of the first equality in (11) and defines path curves (similarity curves) in (x, t)—space. The dependence of F on η involves an arbitrary function $f(\eta)$ which is the solution to some ordinary differential equation obtained by substituting (1) into (3).

Now in order to find out which infinitesimal transformations can be admitted we need also to study the invariance of the differential operator $\mathfrak{M}$. We now calculate how derivatives transform. It is a little more convenient to calculate the operator in coordinates (x', t'). We are interested in partial derivatives along a surface

$$u = \theta(x, t). \tag{13}$$

Along such a surface the general form of (9) is

$$\begin{Bmatrix} x' = x'(x, t) \\ t' = t'(x, t) \end{Bmatrix}. \tag{14}$$

We compute directly how derivatives transform:

$$\begin{aligned} \frac{\partial x}{\partial x'} &= 1 - \epsilon[X_x + X_u \theta_x] + O(\epsilon^2) \\ \frac{\partial x}{\partial t'} &= -\epsilon[X_t + X_u \theta_t] + O(\epsilon^2) \\ \frac{\partial t}{\partial t'} &= 1 - \epsilon[T_t + T_u \theta_t] + O(\epsilon^2) \\ \frac{\partial t}{\partial x'} &= -\epsilon[T_x + T_u \theta_x] + O(\epsilon^2). \end{aligned} \tag{15}$$

Thus we can calculate the transformation between various partial derivatives.

We start from (9) and write

$$\theta'(x', t') = \theta(x, t) + \epsilon U(x, t, \theta) + O(\epsilon^2). \tag{16}$$

Then after collecting terms we have:

$$\frac{\partial \theta'}{\partial x'} = \theta_x + \epsilon[U_x + (U_u - X_x)\theta_x - T_x\theta_t - X_u\theta_x^2 - T_u\theta_x\theta_t] + O(\epsilon^2). \tag{17}$$

The second derivative follows:

$$\begin{aligned}\frac{\partial^2 \theta'}{\partial x'^2} = \theta_{xx} &+ \epsilon[U_{xx} + (2U_{xu} - X_{xx})\theta_x - T_{xx}\theta_t + (U_{uu} - 2X_{xu})\theta_x^2 \\ &- T_{xu}\theta_x\theta_t - X_{uu}\theta_x^3 - T_{uu}\theta_x^2\theta_t + (U_u - 2X_x)\theta_{xx} - 2T_x\theta_{xt} \\ &- 3X_u\theta_{xx}\theta_x - T_u\theta_{xx}\theta_t - 2T_u\theta_{xt}\theta_x] + O(\epsilon^2).\end{aligned} \tag{18}$$

Similar expressions are formed for the time derivatives by interchanging the roles of x and t. For instance

$$\frac{\partial \theta'}{\partial t'} = \theta_t + \epsilon[U_t + (U_u - T_t)\theta_t - X_t\theta_x - T_u\theta_t^2 - X_u\theta_x\theta_t] + O(\epsilon^2). \tag{19}$$

3. General similarity solution of the heat equation. We now apply the results of the previous section to a case which allows many different transformations, the classical heat equation. We try to keep the discussion as general as possible here. For the classical heat equation ($\mathfrak{M}[u] = \partial u/\partial t - \partial^2 u/\partial x^2$ in (3)). We are looking for those infinitesimals (X, T, U) for which the fact that $\theta(x, t)$ is a solution of the heat equation

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} = 0 \tag{20}$$

implies that $\theta'(x', t')$ is also a solution of the heat equation

$$\frac{\partial \theta'}{\partial t'} - \frac{\partial^2 \theta'}{\partial x'^2} = 0. \tag{21}$$

This fact together with the invariance condition (7), which must be investigated, will allow the conclusion (8) that the solution is invariant. From (18) and (19) we have an expression for the heat operator on θ' in terms of the solution $\theta(x, t)$:

$$\begin{aligned}\frac{\partial \theta'}{\partial t'} - \frac{\partial^2 \theta'}{\partial x'^2} = \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} &+ \epsilon[U_t - U_{xx} + (X_{xx} - X_t - 2U_{xu})\theta_x \\ &+ (T_{xx} + U_u - T_t)\theta_t + (2X_{xu} - U_{uu})\theta_x^2 + (2T_{xu} - X_u)\theta_x\theta_t \\ &- T_u\theta_t^2 + X_{uu}\theta_x^3 + T_{uu}\theta_x^2\theta_t + (2X_x - U_u)\theta_{xx} \\ &+ 2T_x\theta_{xt} + 3X_u\theta_{xx}\theta_x + T_u\theta_{xx}\theta_t + 2T_u\theta_{xt}\theta_x] + O(\epsilon^2).\end{aligned} \tag{22}$$

Our first method of proceeding which we call the "classical" method only makes use of the given equation (20) and thus involves setting the right hand side of (22) proportional to $(\theta_t - \theta_{xx})$. This provides a set of conditions on X, T, U without the use of the invariant surface condition (10). Here the latter condition is used later to find the functional form of the solution.

The classical method simply involves equating to zero terms with the same derivatives of θ, *i.e.*, the coefficients of $\theta_x\theta_{tx}$, $\theta_t\theta_x$, θ_x^2, θ_{tx}, θ_t, θ_x and the terms free of derivatives of θ, after substituting θ_t for θ_{xx} in (22). This procedure is sufficient for finding similarity solutions. Successively equating to zero the coefficients of $\theta_x\theta_{tx}$, $\theta_t\theta_x$, $\theta_x\theta_x$ in (22), we find that:

$$\left\{\begin{aligned} T_u &= 0 \\ X_u &= 0 \\ U_{uu} &= 0 \end{aligned}\right\} \tag{23}$$

and thus

$$\left\{\begin{aligned} U(x, t, u) &= f(x, t)u + g(x, t) \\ X(x, t, u) &= X(x, t) \\ T(x, t, u) &= T(x, t) \end{aligned}\right\}. \tag{24}$$

Then successively equating to zero the coefficients of θ_{tx}, θ_t, θ_x, and the remaining terms, we find that:

$$T_x = 0 \quad \text{which implies that} \quad T = T(t) \tag{25}$$

$$2X_x - T'(t) = 0 \tag{26}$$

$$X_t - X_{xx} + 2f_x = 0 \tag{27}$$

$$f_{xx} - f_t = 0 \tag{28}$$

$$g_{xx} - g_t = 0. \tag{29}$$

Thus $g(x, t)$ is any solution to (20). At first we shall only consider the subgroup for which $g(x, t) = 0$.

Solving (26) for X we are led to:

$$X = \frac{xT'(t)}{2} + A(t) \quad \text{with arbitrary} \quad A(t). \tag{26$'$}$$

Substituting (26′) into (27) and solving for f we obtain:

$$f = -\frac{x^2T''(t)}{8} - \frac{xA'(t)}{2} + B(t) \tag{27$'$}$$

where $B(t)$ is arbitrary.

Substitution of (27′) in (28) yields:

$$\frac{-T''(t)}{4} + \frac{x^2T'''(t)}{8} + \frac{xA''(t)}{2} - B'(t) = 0. \tag{28$'$}$$

Solving (28′) we obtain finally the classical group of the heat equation:

$$\left\{\begin{aligned} X &= \kappa + \delta t + \beta x + \gamma x t \\ T &= \alpha + 2\beta t + \gamma t^2 \\ f &= -\gamma\left[\frac{x^2}{4} + \frac{t}{2}\right] - \frac{\delta x}{2} + \lambda \end{aligned}\right\} \tag{30}$$

where α, β, γ, δ, κ, λ are 6 arbitrary parameters.

In (x, t) space, the group (30) is a subgroup of the projective group. All of the parameters, except for γ, individually represent "trivial" transformations. κ represents translation invariance in x, α translations in t, δ represents invariance under a Galilean transformation, and β represents similitudinous invariance which is the invariance used to find the well-known source solution of the heat equation. We will consider those cases where $\gamma \neq 0$.

From (11) we see that if $g(x, t) = 0$, then only the ratios

$$\frac{X}{T} = A(x, t), \qquad \frac{U}{T} = uC(x, t) \tag{31}$$

are needed. Keeping in evidence the translation invariance $(x - x_0, t - t_0)$, and appropriately relabelling the constants, we can rewrite (30) as

$$A(x, t) = \frac{(x - x_0)(t - t_0) + Va_0}{a_0 + (t - t_0)^2}. \tag{32}$$

Correspondingly, for $C(x, t)$, we may write

$$C(x, t) = \frac{\mu - \frac{1}{2}(t - t_0) - \frac{1}{4}(x - x_0)^2}{a_0 + (t - t_0)^2}. \tag{33}$$

We now find the similarity curves by integrating

$$\frac{dx}{dt} = A(x, t). \tag{34}$$

It is convenient to treat the cases $a_0 > 0$, $a_0 < 0$ separately since the integrals which occur and the geometry of the path curves are different.

For

$$a_0 > 0, \quad \text{let} \quad a_0 = a^2$$
$$a_0 < 0, \quad \text{let} \quad a_0 = -b^2.$$

We show details for $a_0 < 0$; write (34) as

$$\frac{dx}{dt} = \frac{xt - Vb^2}{t^2 - b^2}. \tag{35}$$

We can always use the correspondence

$$x \leftrightarrow x - x_0, \qquad t \leftrightarrow t - t_0. \tag{36}$$

This linear equation can be written as

$$\frac{d}{dt}\frac{x}{(t^2 - b^2)^{1/2}} = -\frac{Vb^2}{(t^2 - b^2)^{3/2}} \tag{37}$$

so that integration gives

$$\frac{x}{(t^2 - b^2)^{1/2}} = \eta + V\frac{t}{(t^2 - b^2)^{1/2}}. \tag{38}$$

η is the constant of integration and $\eta(x, t) = \text{const.}$ are the equations of the similarity curves. Thus

$$\eta = \frac{(x - x_0) - V(t - t_0)}{((t - t_0)^2 - b^2)^{1/2}} \tag{39}$$

or

$$x - x_0 = V(t - t_0) + \eta((t - t_0)^2 - b^2)^{1/2}. \tag{40}$$

For $a_0 > 0$, we have correspondingly

$$x - x_0 = V(t - t_0) + \eta((t - t_0)^2 + a^2)^{1/2}. \tag{41}$$

These curves have 4 parameters (x_0 , t_0 , V, a). Sketches of the similarity curves are shown in Figs. 2, 3. The curves of (20) are evidently well adapted for initial value problems since $t = t_0 + \beta$ is a similarity curve. Several limiting cases are of interest since simplified equations result.

Limiting Case 1. $b^2 \to 0$. This case is between the two cases illustrated in

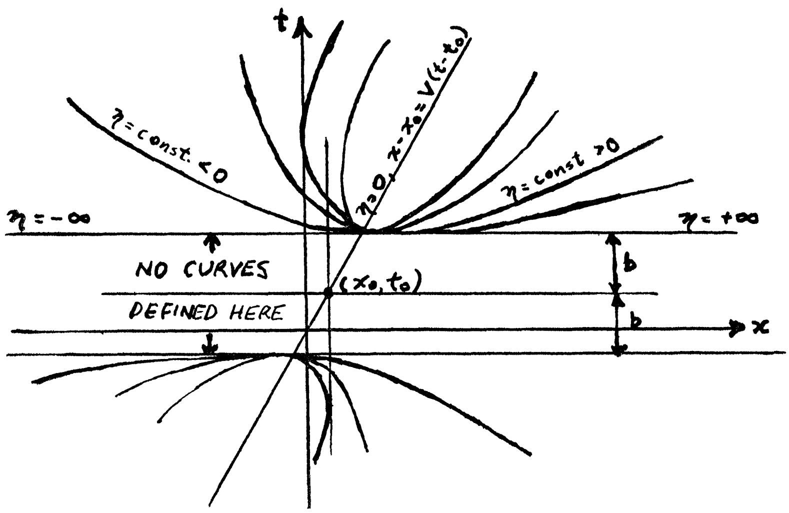

Fig. 2 Similarity Curves for $a_0 < 0$

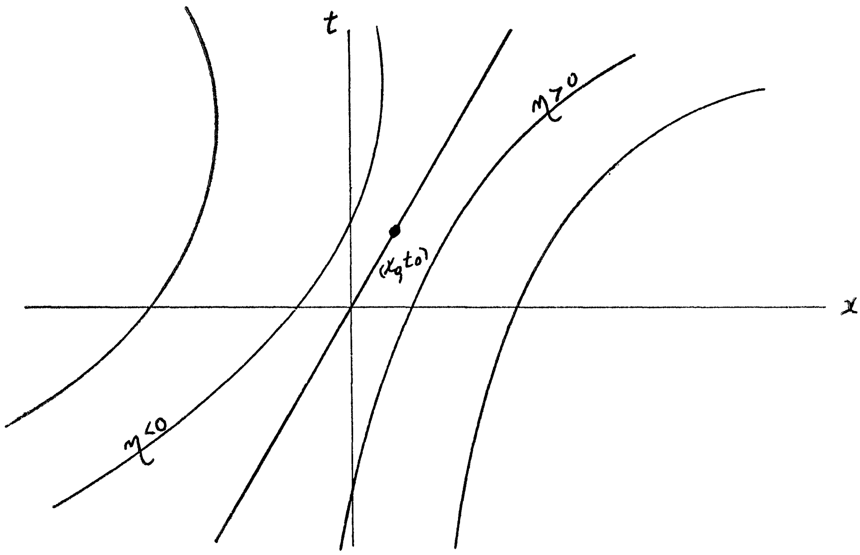

Fig. 3 Similarity Curves for $a_0 > 0$

Figs. 2, 3. For $t > t_0 + b$ we have in (40) as $b^2 \to 0$

$$x - x_0 = V(t - t_0) + \eta(t - t_0)\left\{1 - \frac{1}{2}\frac{b^2}{(t - t_0)^2} + \cdots\right\}. \tag{42}$$

A distinguished case occurs if as $b \to 0$, $\eta \to \infty$ so that b^2 is finite. This limit only makes sense if V gets large and η is relabeled.

$$\eta = -V + \eta^* \quad \text{where} \quad Vb^2 = -2k \tag{43}$$

and consider the limit to take place with k fixed. (42) becomes

$$x - x_0 = \eta^*(t - t_0) - \frac{k}{(t - t_0)} \tag{44}$$

or

$$\eta^* = \frac{x - x_0}{t - t_0} + \frac{k}{(t - t_0)^2}. \tag{45}$$

The similarity curves $\eta^* =$ const. have 3 parameters (x_0, t_0, k). Sketches appear below.

Limiting Case 2. $a^2 \to \infty$: Write (41) as

$$x - x_0 = V(t - t_0) + \eta a\left\{1 + \frac{1}{2}\frac{(t - t_0)^2}{a^2} + \cdots\right\}. \tag{46}$$

Then as $a \to \infty$ we need $\eta \to \infty$ to get a significant limit. Thus also $x_0 \to \infty$. Let

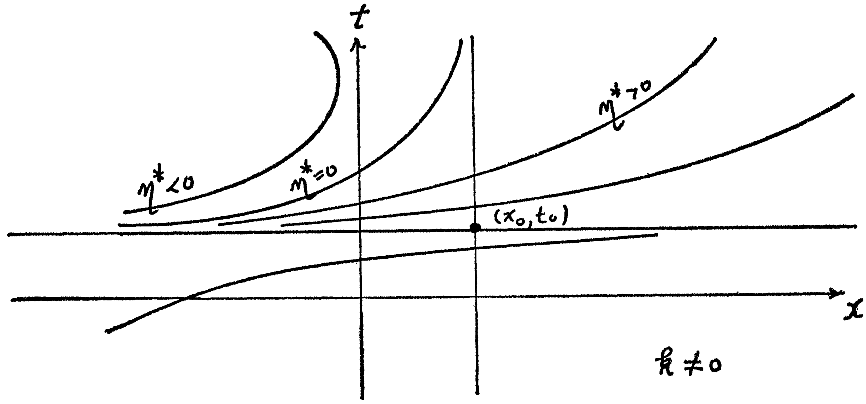

FIG. 4

$$\eta = 2ka - \frac{\eta^*}{a} \tag{47}$$

$$x_0 = -2ka^2 + x_1 \tag{48}$$

and consider (k, η^*, x_0, t_1) fixed as $a \to \infty$. Then (46) becomes

$$\eta^* = V(t - t_0) + k(t - t_0)^2 - (x - x_1). \tag{49}$$

Here the similarity curves $\eta^* =$ const. have 4 parameters (x_1, t_0, V, k) and are shown in Fig. 6.

Next the functional form of the solution corresponding to the general similarity curves (39) and the special limiting cases (45) and (49) is worked out. In order to do this, we integrate

$$\frac{d\theta}{\theta} = C(x(\eta, t), t)\, dt. \tag{50}$$

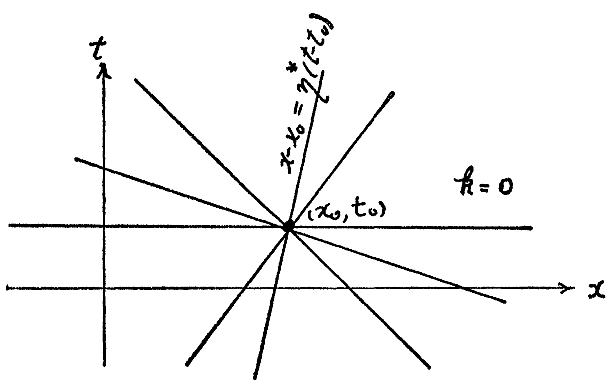

FIG. 5

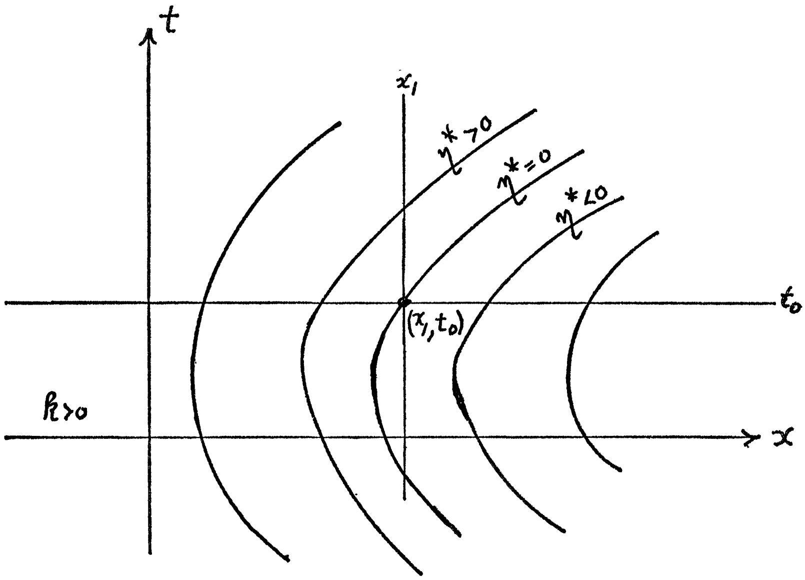

FIG. 6

Using (40) and (33) $[(x - x_0) \leftrightarrow x,\ (t - t_0) \leftrightarrow t]$ we have

$$(51) \qquad C(x(\eta, t), t)$$

$$= -\frac{1}{4}\eta^2 + \frac{\mu}{t^2 - b^2} - \frac{1}{2}\frac{t}{t^2 - b^2} - \frac{1}{4}V^2\frac{t^2}{t^2 - b^2} - \frac{1}{2}\eta V\frac{t}{(t^2 - b^2)^{1/2}}.$$

Integration of these terms is elementary:

$$(52) \qquad \log\theta = \int C\,dt = -\frac{1}{4}\eta^2 t - \frac{\mu}{2b}\log\left(\frac{t + b}{t - b}\right) - \frac{1}{4}\log(t^2 - b^2)$$

$$+ \frac{1}{4}V^2\left[-t + \frac{b}{2}\log\left(\frac{t + b}{t - b}\right)\right] - \frac{1}{2}\eta V(t^2 - b^2)^{1/2} + \log F(\eta).$$

Then the general similarity solution for this case has the form

$$(53) \qquad \theta(x, t) = \frac{1}{[(t - t_0)^2 - b^2]^{1/4}}\left[\frac{(t - t_0) + b}{(t - t_0) - b}\right]^{\rho}$$

$$\cdot[\exp\{-\tfrac{1}{4}(\eta^2 - V^2)(t - t_0) - \tfrac{1}{2}V(x - x_0)\}]F(\eta)$$

where

$$\rho = \frac{1}{8}bV^2 - \frac{1}{2}\frac{\mu}{b}, \qquad \eta = \frac{(x - x_0) - V(t - t_0)}{((t - t_0)^2 - b^2)^{1/2}}.$$

The independent parameters in this solution are (x_0, t_0, V, b, μ). The solution needs a slight modification in the complementary case $(a_0 > 0)$. In (51) we can replace b^2 by $-a^2$ and then $\int dt/(t^2 + a^2)$ contributes $\tan^{-1} t/a$ terms. Thus, instead of (53), we obtain

$$\theta(x, t) = \frac{1}{[(t - t_0)^2 + a^2]^{1/4}} \left[\exp \left\{\left(\frac{a}{4} V^2 + \frac{\mu}{a}\right) \tan^{-1} \left(\frac{t - t_0}{a}\right) - \tfrac{1}{4}(\eta^2 - V^2)(t - t_0) - \tfrac{1}{2}V(x - x_0)\right\}\right] F(\eta) \tag{54}$$

where

$$\eta = \frac{(x - x_0) - V(t - t_0)}{((t - t_0)^2 + a^2)^{1/2}}.$$

Now, the ordinary differential equation satisfied by $F(\eta)$, must be found by substituting the similarity forms (53) or (54) into the basic heat equation (20). The result is

$$\frac{d^2F}{d\eta^2} + \left(\frac{b^2 V^2}{4} - \mu - \frac{b^2\eta^2}{4}\right)F = 0. \tag{55}$$

Corresponding to (54) replace b^2 by $-a^2$. The differential equation is that of the parabolic cylinder functions (a special case of confluent hypergeometric functions) and the solutions can be expressed in standard forms corresponding to the equation

$$\frac{d^2F}{dz^2} + (\nu + \tfrac{1}{2} - \tfrac{1}{4}z^2)F = 0 \tag{56}$$

where

$$z = b^{1/2}\eta, \qquad \nu + \tfrac{1}{2} = \tfrac{1}{4}bV^2 - \mu/b.$$

Any two of

$$D_\nu(z),\ D_\nu(-z),\ D_{-\nu-1}(iz),\ D_{-\nu-1}(-iz)$$

are linearly independent solutions of (56). Their properties are well known.

$$D_\nu(z) = 2^{1/2\nu} e^{-z^2/4} \cdot \left[\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{1}{2}\nu)}\, {}_1F_1(-\tfrac{1}{2}\nu; \tfrac{1}{2}; \tfrac{1}{2}z^2) + 2^{-1/2}z\, \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}\nu)}\, {}_1F_1(\tfrac{1}{2} - \tfrac{1}{2}\nu; \tfrac{3}{2}; \tfrac{1}{2}z^2)\right]. \tag{57}$$

For integer values $\nu = n = 0, 1, 2$ the solutions are expressed by (orthogonal) Hermite polynomials

$$\begin{aligned} D_n(z) &= e^{-z^2/4}\, \mathrm{He}_n(z); \qquad \mathrm{He}_n = (-1)^n e^{z^2/2} \frac{d^n}{dz^n} e^{-z^2/2} \\ &\qquad \mathrm{He}_0 = 1, \quad \mathrm{He}_1 = z, \quad \mathrm{He}_2 = z^2 - 1, \quad \text{etc.} \end{aligned} \tag{58}$$

while

$$D_{-1}(z) = e^{z^2/4}(2\pi)^{-1/2} \operatorname{erfc}(2^{-1/2}z), \qquad D_{-2} = \text{etc.} \tag{59}$$

The series (57) shows the behavior for small z while asymptotically for $|z| \gg |\nu|, 1$

$$D_\nu(z) = z^\nu e^{-z^2/4}[1 + O(z^{-2})], \qquad |\arg z| < \frac{3\pi}{4}$$

(60)
$$= z^\nu e^{-z^2/4}[1 + O(z^{-2})] - \frac{(2\pi)^{1/2}}{\Gamma(-\nu)} e^{\nu\pi i} z^{-\nu-1} e^{z^2/4}[1 + O(z^{-2})],$$
$$\frac{5\pi}{4} > \arg z > \tfrac{1}{4}\pi$$
$$= z^\nu e^{-z^2/4}[1 + O(z^{-2})] - \frac{(2\pi)^{1/2}}{\Gamma(-\nu)} e^{-\nu\pi i} z^{-\nu-1} e^{z^2/4}[1 + O(z^{-2})],$$
$$-\tfrac{1}{4}\pi > \arg z > -\frac{5\pi}{4}.$$

The parabolic cylinder functions are *entire* functions of z.

According to the similarity of a particular problem a choice is made for the solutions F. The boundary conditions are prescribed on a similarity curve $\eta = \eta_0 =$ const. and $F(\eta_0)$ (or F', or a linear combination of F' and F) must be chosen accordingly (*cf.* 53). One point is worth mentioning which applies to various linear cases. The parameter μ appears only in the equation and not in the similarity coordinate or boundary condition. Thus μ can play the role of an eigenvalue in the case of homogeneous boundary conditions; superposition of eigenfunctions can be used to represent arbitrary conditions on a suitably chosen curve where $\eta \neq$ const.

Special Case. ($b \to \infty$, $t_0 \to -\infty$, $V \to 0$, $\mu \to \infty$): To connect up with the well known fundamental solution which is expressed in $(xt^{-1/2})$ it is necessary to consider a limiting case of (53). Let

(61)
$$b = t_0 - t_1$$

and consider the limit where $t_0 \to -\infty$, t_1 fixed, $Vt_0 \to 0$. Then

$$(t - t_0)^2 - b^2 = t^2 - 2tt_0 + t_0^2 - (t_0 - t_1)^2 \to (-2t_0)(t - t_1)$$

and

(62)
$$\eta = \frac{(x - x_0) - V(t - t_0)}{((t - t_0)^2 - b^2)^{1/2}} \to \frac{x - x_0}{(-2t_0(t - t_1))^{1/2}}.$$

Let

(63)
$$\bar{\eta} = \eta(-2t_0)^{1/2} = \frac{x - x_0}{(t - t_1)^{1/2}}.$$

Then, disregarding the infinite multiplication constant (which can always be scaled out), the general similarity solution (53) takes the form

$$\theta(x, t) = \frac{1}{(t - t_1)^{\omega/2+1/4}} e^{-\bar{\eta}^2/8} F(\bar{\eta}). \tag{64}$$

We have assumed $\mu \to \infty$ so that

$$\rho = \frac{bV^2}{8} - \frac{1}{2}\frac{\mu}{b} \to -\frac{1}{2}\frac{\mu}{t_0} \to -\frac{1}{2}\omega, \qquad \omega = \text{const}.$$

Then, the differential equation (55) becomes

$$\frac{d^2F}{d\bar{\eta}^2} + \left(\frac{1}{2}\omega - \frac{\bar{\eta}^2}{16}\right)F(\bar{\eta}) = 0. \tag{65}$$

This is brought to the standard form (56) if

$$z = \frac{\bar{\eta}}{2^{1/2}} = \frac{x - x_0}{(2(t - t_1))^{1/2}}, \qquad \omega = \nu + \tfrac{1}{2}. \tag{66}$$

Then (53) becomes

$$\theta(x, t) = \frac{1}{(t - t_1)^{(\nu+1)/2}} e^{-z^2/2} F(z). \tag{67}$$

If the initial line $t = 0$ is to be a similarity curve we must choose $t_1 = 0$. For the source problem we must have a solution vanishing (actually exponentially) as $z \to \pm\infty$. The asymptotic form (60) shows that integer values of ν are necessary while $\int_{-\infty}^{\infty} u\, dx = \text{const.}$ implies $\nu = 0$. Then the solution (67) becomes (the constant is chosen from overall conservation of heat):

$$\theta(x, t) = (4\pi t)^{-1/2} e^{-x^2/4t} \tag{68}$$

Other well known solutions are included in (67). For example, if $\theta(x, 0) = 0$, $\theta(0, t) = 1 (x > 0, t > 0)$ then $\nu = -1$ and

$$\theta(x, t) = e^{-z^2/4} C \cdot D_{-1}(z) = \operatorname{erfc} \frac{z}{2^{1/2}} = \operatorname{erfc} \frac{x}{2t^{1/2}}. \tag{69}$$

Of course, in general in $x > 0$, $t > 0$ with zero initial conditions (67) gives

$$\theta(x, t) = \frac{C}{t^{(\nu+1)/2}} e^{-z^2/4} D_\nu(z) \tag{70}$$

and at $x = 0 (z = 0)$ we have

$$\theta(0, t) = \frac{C}{t^{(\nu+1)/2}} D_\nu(0) = C \frac{\Gamma(\frac{1}{2}) 2^{1/2\nu}}{\Gamma(\frac{1}{2} - \frac{1}{2}\nu) t^{(\nu+1)/2}}. \tag{71}$$

As $t \to 0$, the solution makes physical sense only for $\nu < 0$, so that a finite amount of heat is added across the boundary $x = 0$. Such considerations always have to be added and are never contained in the similarity reasoning.

Next we construct the functional form of the solution and find the differential equations for the two limiting cases discussed earlier. The limit processes could of course be applied to the solution but it is simpler to apply them to the equations.

Limiting Case 1. $b^2 \to 0 (Vb^2 = -2k)$: The limit process is that of (43) with the result (45) and the similarity curves of Fig. 5. We carry out the limit on the general solution (53).

$$\theta(x, t) \to (t - t_0)^{-1/2} \cdot [\exp \{-\tfrac{1}{4}\eta^{*2}(t - t_0) + k^2/3(t - t_0)^3 - \mu/(t - t_0)\}]F(\eta^*) \tag{72}$$

where

$$\eta^* = \frac{x - x_0}{t - t_0} + \frac{k}{(t - t_0)^2}.$$

In the equation for $F(\eta)$, (55), the same limit is performed and (53) becomes

$$\frac{d^2F}{d\eta^{*2}} - (\mu + k\eta^*)F(\eta^*) = 0. \tag{73}$$

Limiting Case 2. $a^2 \to \infty$: We use the solution form (54) replacing μ by μ^*. From equation (55),

$$\frac{d^2F}{d\eta^2} - (\tfrac{1}{4}a^2V^2 - \tfrac{1}{4}a^2\eta^2 + \mu^*)F = 0$$

we obtain

$$\frac{d^2F}{d\eta^{*2}} + (k^2a^2 - k\eta^* - \tfrac{1}{4}V^2/a^2 - \mu^*/a^2)F = 0 \tag{74}$$

and keeping V fixed we let

$$\mu^* = k^2a^4 + \mu a^2 + \cdots \quad \text{as} \quad a \to \infty. \tag{75}$$

Thus

$$\frac{d^2F}{d\eta^{*2}} - (k\eta^* + \mu)F(\eta^*) = 0 \tag{76}$$

which is the same equation as (73). With the further simplifications of (75) and omitting the scale factor, we find that

$$\theta(x, t) = (\exp \{[k\eta^* + \tfrac{1}{2}V^2 + \mu](t - t_0) - \tfrac{1}{3}k^2(t - t_0)^3 - \tfrac{1}{2}V(x - x_1)\})F(\eta^*) \tag{77}$$
$$\eta^* = -[(x - x_1) - V(t - t_0) - k(t - t_0)^2].$$

A special case of the solutions (77) have been used by J. M. Burgers [4] as eigensolutions of the heat equation for a semi-infinite space bounded by a moving boundary of parabolic form

$$x = \tfrac{1}{2}t^2 \tag{78}$$

(See Fig. 7). These eigensolutions are obtained from (77) if

$$k = \tfrac{1}{2}, \qquad V = 0, \qquad x_1 = 0, \qquad t_0 = 0 \tag{79}$$

and

$$\mu = -\sigma_m \tag{80}$$

is the eigenvalue. Eqn. (77) becomes

$$\begin{gathered}\theta_m(x, t) = (\exp\{-\tfrac{1}{12}t^3 + \tfrac{1}{2}\eta^* t - \sigma_m t\})F_m(\eta^*) \\ \eta^* = \tfrac{1}{2}t^2 - x.\end{gathered} \tag{81}$$

The $F_m(\eta^*)$ are the Airy function solutions of (76) which die out as $\eta^* \to \infty$

$$\frac{d^2F_m}{d\eta^{*2}} - (\tfrac{1}{2}\eta^* - \sigma_m)F_m = 0 \tag{82}$$

$$F_m(\eta^*) = \frac{(\eta^* - 2\sigma_m)^{1/2}}{2^{1/6}3^{1/3}} K_{1/3}\left(\frac{2^{1/2}}{3}[\eta^* - 2\sigma_m]^{3/2}\right). \tag{83}$$

The σ_m are chosen so that $(-2\sigma_m)$ are the zeroes of the Airy function or of the continuation of (83) to negative arguments

$$\begin{aligned}F_m(\eta^*) = {}& \frac{\pi(2\sigma_m - \eta^*)^{1/2}}{2^{1/6}3^{5/6}} \\ & \cdot\left\{J_{-1/3}\left(\frac{2^{1/2}}{3}(2\sigma_m - \eta^*)^{3/2}\right) + J_{1/3}\left(\frac{2^{1/2}}{3}(2\sigma_m - \eta^*)^{3/2}\right)\right\}.\end{aligned} \tag{84}$$

Thus $F_m(0) = 0$, all m. Burgers shows that this set of eigenfunctions is orthogonal and thus uses them to construct a representation of a unit source at (x_s, t_s) reflected in the moving boundary. This representation is

$$\begin{aligned}\theta(x, t; x_s, t_s) = {}& \tfrac{1}{2}\exp\left\{-\frac{t^3 - t_s^3}{12} + \frac{t\eta^* - t_s\eta_s^*}{2}\right\} \\ & \cdot\sum_{m=0}^{\infty}\frac{F_m(\eta^*)F_m(\eta_s^*)}{F_m'(0)^2}e^{-\sigma_m(t-t_s)}, \qquad t \geqq t_s, \qquad \theta = 0, \qquad t \leqq t_s.\end{aligned} \tag{85}$$

Burgers' method of deriving (81) was to assume a separation of variables form $G(t, v)\, H(t, v)$ and to choose G so that essentially a similarity form resulted for H_0.

Next we consider the influence of $g(x, t)$ (*cf.* 29) and take as an example the "trivial" case where

$$g(x, t) = c = \text{const.}$$

and

$$\left\{\begin{aligned}X &= x \\ T &= 2t\end{aligned}\right\}. \tag{86}$$

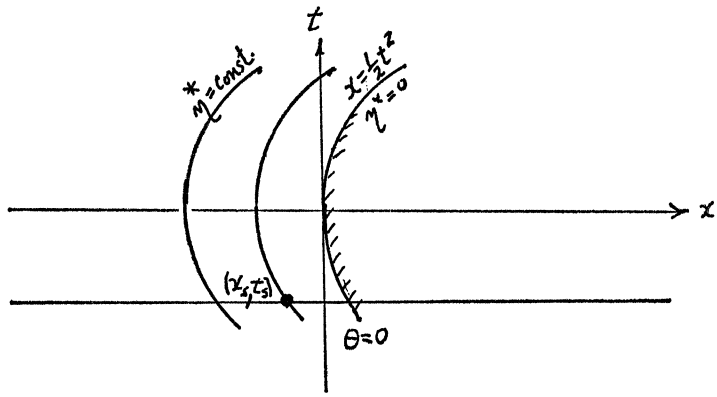

FIG. 7

Here the similarity variable is $\eta = x/t^{1/2}$ and

(87) $$\theta = c \log t^{1/2} + F(\eta).$$

Thus, for this linear case, the effect of $g(x, t) \neq 0$ is to make $F(\eta)$ the solution of an inhomogeneous equation. Then $F(\eta)$ satisfies

(88) $$F'' + \tfrac{1}{2}\eta F' = \tfrac{1}{2}c.$$

Thus

(89) $$F' = \tfrac{1}{2}ce^{-\eta^2/4} \int_\alpha^\eta e^{\eta^2/4}\, d\eta.$$

Next we give the "non-classical" method of procedure which makes use of both the given equation (20) and the invariant surface condition (10).

Since in (10) the equation can be divided through by (X, T, or U) there are really only two independent infinitesimals. Thus we will assume that $T \neq 0$, divide through by T, and thus in (10) replace

(90) $$\frac{X}{T} \to X, \qquad \frac{U}{T} \to U.$$

The condition for an invariant surface now reads

(91) $$\theta_t = U - X\theta_x$$

which implies that

(92) $$\theta_{tx} = (U_x - XU) + (U_u - X_x + X^2)\theta_x - X_u\theta_x^2\,.$$

Now using (91), (92) the RHS of (22) can be expressed as coefficients times powers of θ_x. Successively equating to zero the coefficients of θ_x^3, θ_x^2, θ_x^1, and θ_x^0 we get:

(93) $$X_{uu} = 0$$

$$U_{uu} - 2X_{xu} + 2XX_u = 0 \tag{94}$$

$$X_t - 2UX_u + 2U_{xu} - X_{xx} + 2XX_x = 0 \tag{95}$$

$$U_{xx} - 2UX_x - U_t = 0. \tag{96}$$

If we differentiate (94) once and (95) thrice with respect to u, then we have:

$$\begin{aligned} U_{uuu} + 2X_uX_u &= 0 \\ U_{uuu}X_u &= 0. \end{aligned} \tag{97}$$

Then using (94) again we find that:

$$\begin{Bmatrix} U_{uu} = 0 \\ X_u = 0 \end{Bmatrix} \tag{98}$$

and thus

$$\begin{Bmatrix} U = uC(x, t) + D(x, t) \\ X = A(x, t) \end{Bmatrix}. \tag{99}$$

From (95) we have

$$A_t + 2AA_x - A_{xx} = -2C_x \tag{100}$$

and from (96)

$$C_t - C_{xx} + 2A_xC = 0 \tag{101}$$

$$D_t - D_{xx} + 2A_xD = 0. \tag{102}$$

The characteristic differential equations corresponding to (91) are now

$$\frac{dx}{A(x, t)} = \frac{dt}{1} = \frac{d\theta}{D(x, t) + C(x, t)\theta}. \tag{103}$$

The similarity variable

$$\eta(x, t) = \text{const.} \tag{104}$$

is the integral of the first two of (103) and defines the similarity curves. Once $\eta(x, t)$ is known explicitly the functional form is found, for example, by replacing x by $x(t, \eta)$ and integrating the second of (103).

The problem of finding the general similarity solution to the heat equation has thus been "reduced" to the study of the nonlinear equations (100), (101), and (102), C and A are coupled and $D \sim C$ is always possible. However, it is clearly impossible to construct the general solution of these equations. Rather classes of special solutions must be examined, each of which generates a similarity solution of the original heat equation. But *any* solution to the system (100, 101, 102) reduces the heat equation to an ordinary differential equation.

Let $A = 2\varphi_x$, then from (100),

$$C = \varphi_{xx} - \varphi_t - 2\varphi_x^2 + M(t). \tag{105}$$

If $M(t) = 0$, then (101) implies that

$$\varphi_{xxxx} - 2\varphi_{txx} - 4\varphi_x\varphi_{xxx} - 8\varphi_{xx}\varphi_{xx} + 4\varphi_{xx}\varphi_t + 8\varphi_{xx}\varphi_x^2 + \varphi_{tt} + 4\varphi_x\varphi_{xt} = 0. \tag{106}$$

The "classical" case results when we set $\varphi_{xxx} = 0(A_{xx} = 0)$.

It is interesting to note that if $C = 0$, then (101) becomes the Burgers equation. Setting $A = -H_x/H$, where H is a solution to the heat equation, we can show that the resulting solution θ satisfies the relation $\theta_x = H$.

The non-classical solutions are more general than the classical ones. However, all simple solutions we have thus far found for this case are included in the classical case. For other equations the non-classical solutions have been shown to be more general.

4. Conclusion. The main aim of this paper has been to demonstrate the use of the method of infinitesimal transformations for the construction of similarity solutions of partial differential equations. The general forms of the similarity solution (53) and (54) are new results and contain many new special cases. In addition the "non-classical" method represents a new approach to the discovery of similarity solutions.

The same method can be applied to a wide variety of linear and nonlinear cases. For example, the system of equations for one-dimensional unsteady gasdynamics is discussed in [2]. In [5], the method is applied to a Fokker–Planck equation, a general axi-symmetric wave equation, and a nonlinear heat equation.

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