

# Construction of conservation laws: how the direct method generalizes Noether's theorem

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This paper shows how to construct directly the local conservation laws for essentially any given DE system. This comprehensive treatment is based on first finding conservation law multipliers. It is clearly shown how this treatment is related to and subsumes the classical Noether's theorem (which only holds for variational systems). In particular, multipliers are symmetries of a given PDE system only when the system is variational as written. The work presented in this paper amplifies and clarifies earlier work by the first and third authors.

## 1 Introduction

A conservation law of a non-degenerate DE system is a divergence expression that vanishes on all solutions of the DE system. In general, any such nontrivial expression that yields a local conservation law of a given DE system arises from a linear combination formed by *local multipliers (characteristics)* with each DE in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of derivatives of the dependent variables of the given DE system. It turns out that a divergence expression depending on independent variables, dependent variables and their derivatives to some finite order is annihilated by the Euler operators associated with each of its dependent variables; conversely, if the Euler operators, associated with each dependent variable in an expression involving independent variables, dependent variables and their derivatives to some finite order, annihilate the expression, then the expression is a divergence expression. From this it follows that a given DE system has a local conservation law if and only if there exists a set of local multipliers whose scalar product with each DE in the system is identically annihilated *without restricting* the dependent variables in the scalar product to solutions of the DE system, i.e., the dependent variables, as well as each of their derivatives, are treated as *arbitrary* functions.

Thus the problem of finding local conservation laws of a given DE system reduces to the problem of finding sets of local multipliers whose scalar product with each DE in the system is annihilated by the Euler operators associated with each dependent variable where the dependent variables and their derivatives in the given DE system are replaced by arbitrary functions. Each such set of local multipliers yields a local conservation law of the given DE system. Moreover, for any given set of local conservation law multipliers, there is an integral formula to obtain the fluxes of the local conservation law [1–3]. Often it is straightforward to obtain the conservation law by direct calculation after its multipliers are known [4]. What has been outlined here is the *direct method* for obtaining local conservation laws.

For a given DE system, Lie’s algorithm yields an over-determined set of linear determining equations whose solutions yield local symmetries. This set of linear PDEs arises from the linearization of the given DE system (*Fréchet derivative*) about an *arbitrary* solution of the given DE system, i.e., the resulting set of linear PDEs must hold for each solution of the given DE system. After the given DE system and its differential consequences are substituted into its linearization, the resulting linear PDE system yielding local symmetries must hold with the remaining dependent variables and their derivatives of the given DE system replaced by *arbitrary* functions.

In contrast, for a given DE system, sets of local conservation law multipliers are solutions of an over-determined set of linear determining equations arising from annihilations by Euler operators. It turns out that the set of linear multiplier determining equations for local conservation law multipliers includes the *adjoint* of the set of linear PDEs arising from the linearization of the given PDE system about an arbitrary solution of the given DE system [1].

It follows that in the situation when the set of linearized equations of a given DE system (Fréchet derivative) is self-adjoint, the set of multiplier determining equations includes the set of local symmetry determining equations. Consequently, here each set of local conservation law multipliers yields a local symmetry of the given DE system. In particular, the local conservation law multipliers are also components of the infinitesimal generators of local symmetries in evolutionary form. However, in the self-adjoint case, the set of linear determining equations for local conservation law multipliers is more over-determined than those for local symmetries since here the set of linear determining equations for local conservation law multipliers includes additional linear PDEs as well as the set of linear PDEs for local symmetries. Consequently, in the self-adjoint case, there can exist local symmetries that do not yield local conservation law multipliers.

Noether [5] showed that if a given system of DEs admits a variational principle, then any one-parameter Lie group of point transformations that leaves invariant the action functional yields a local conservation law. In particular, she gave an explicit formula for the fluxes of the local conservation law. Noether’s theorem was extended by Bessel-Hagen [6] to allow the one-parameter Lie group of point transformations to leave invariant the action functional to within a divergence

term. As presented, their results depend on Lie groups of point transformations used in their canonical form, i.e., not in evolutionary form. Boyer [7] showed how all such local conservation laws could be obtained from Lie groups of point transformations used in evolutionary form. From this point of view, it is straightforward to apply Noether's theorem to obtain a local conservation law for any for any one-parameter higher-order local transformation leaving invariant the action functional to within a divergence term. Such a higher-order transformation that leaves invariant an action functional to within a divergence term is called a *variational symmetry*.

As might be expected, Noether's explicit formula for a local conservation law arises from sets of local multipliers that yield components of local symmetries in evolutionary form. From this point of view, it follows that all local conservation laws arising from Noether's theorem are obtained by the direct method. Moreover, one can see that a variational symmetry must map an extremal of the action functional to another extremal. Since an extremal of an action functional is a solution of the DE system arising from the variational principle, it follows that a variational symmetry must be a local symmetry of the given DE system arising from the variational principle.

A system of DEs (as written) has a variational principle if and only if its linearized system (Fréchet derivative) is self-adjoint [8–10]. From this point of view, it also follows that all conservation laws obtained by Noether's theorem must arise from the direct method.

The direct method supersedes Noether's theorem. In particular, for Noether's theorem, including its generalizations by Bessel-Hagen and Boyer, to be directly applicable to a given DE system, the following must hold:

- The linearized system of the given DE system is self-adjoint.
- One has an explicit action functional.
- One has a one-parameter local transformation that leaves the action functional invariant to within a divergence. In order to find such a variational symmetry systematically, one first finds local symmetries (solutions) of the linearized system and then checks whether or not such local symmetries leave the action functional invariant to within a divergence.

On the other hand, the direct method is applicable to any given DE system, whether or not its linearized system is self-adjoint. No functional needs to be determined. Moreover, a set of local conservation law multipliers is represented by *any* solution of an over-determined linear PDE system satisfied by the multipliers and this over-determined linear PDE system is obtained directly from the given DE system. As mentioned above, in the case when the linearized system is self-adjoint, the local symmetry determining equations are a subset of this over-determined linear PDE system.

In the study of DEs, conservation laws have many significant uses. They describe physically conserved quantities such as mass, energy, momentum and an-

gular momentum, as well as charge and other constants of motion. They are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in stability analysis and the global behavior of solutions. In addition, they play an essential role in the development of numerical methods and provide an essential starting point for finding potential variables and nonlocally related systems. In particular, a conservation law is fundamental in studying a given DE in the sense that it holds for any posed data (initial and/or boundary conditions). Moreover, the structure of conservation laws is coordinate-independent since a point (contact) transformation maps a conservation law to a conservation law.

The rest of this paper is organized as follows. In Section 2, the direct method is presented with a nonlinear telegraph system and the Korteweg-de Vries equation used as examples. Noether's theorem is presented in Section 3. In Section 4, there is a discussion of the limitations of Noether's theorem and the consequent advantages of the direct method.

## 2 The direct method

Consider a system  $\mathbf{R}\{x; u\}$  of  $N$  differential equations of order  $k$  with  $n$  independent variables  $x = (x^1, \dots, x^n)$  and  $m$  dependent variables  $u(x) = (u^1(x), \dots, u^m(x))$ , given by

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \dots, \partial^k u) = 0, \quad \sigma = 1, \dots, N. \quad (1)$$

**Definition 2.1.** A *local conservation law* of the DE system (1) is a divergence expression

$$D_i \Phi^i[u] = D_1 \Phi^1[u] + \dots + D_n \Phi^n[u] = 0 \quad (2)$$

holding for all solutions of the DE system (1).

In (2),  $D_i$  and  $\Phi^i[u] = \Phi^i(x, u, \partial u, \dots, \partial^r u)$ ,  $i = 1, \dots, n$ , respectively are total derivative operators and the *fluxes* of the conservation law.

**Definition 2.2.** A DE system  $\mathbf{R}\{x; u\}$  (1) is *non-degenerate* if (1) can be written in Cauchy-Kovalevskaya form [3, 10] after a point (contact) transformation, if necessary.

In general, for a given non-degenerate DE system (1), nontrivial local conservation laws arise from seeking scalar products that involve linear combinations of the equations of the DE system (1) with *multipliers* (factors) that yield nontrivial divergence expressions. In seeking such expressions, the dependent variables and each of their derivatives that appear in the DE system (1) or in the multipliers, are replaced by arbitrary functions. Such divergence expressions vanish on all solutions of the DE system (1) provided the multipliers are non-singular.

In particular a set of multipliers  $\{\Lambda_\sigma[U]\}_{\sigma=1}^N = \{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$  yields a divergence expression for the DE system  $\mathbf{R}\{x; u\}$  (1) if the identity

$$\Lambda_\sigma[U]R^\sigma[U] \equiv D_i\Phi^i[U] \quad (3)$$

holds for *arbitrary* functions  $U(x)$ . Then on the solutions  $U(x) = u(x)$  of the DE system (1), if  $\Lambda_\sigma[u]$  is non-singular, one has a local conservation law

$$\Lambda_\sigma[u]R^\sigma[u] = D_i\Phi^i[u] = 0.$$

[A multiplier  $\Lambda_\sigma[U]$  is *singular* if it is a singular function when computed on solutions  $U(x) = u(x)$  of the given DE system (1) (e.g., if  $\Lambda_\sigma[U] = F[U]/R^\sigma[U]$ ). One is only interested in non-singular sets of multipliers, since the consideration of singular multipliers can lead to arbitrary divergence expressions that are not conservation laws of a given DE system.]

**Definition 2.3.** The *Euler operator* with respect to  $U^\mu$  is the operator defined by

$$E_{U^\mu} = \frac{\partial}{\partial U^\mu} - D_i \frac{\partial}{\partial U^\mu} + \dots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial U_{i_1 \dots i_s}^\mu} + \dots \quad (4)$$

By direct calculation, one can show that the Euler operators (4) annihilate *any* divergence expression  $D_i\Phi^i(x, U, \partial U, \dots, \partial^r U)$  for any  $r$ . In particular, the following identities hold for arbitrary  $U(x)$ :

$$E_{U^\mu}(D_i\Phi^i(x, U, \partial U, \dots, \partial^r U)) \equiv 0, \quad \mu = 1, \dots, m.$$

It is straightforward to show that the converse also holds. Namely, the only scalar expressions annihilated by Euler operators are divergence expressions. This establishes the following theorem.

**Theorem 2.1.** *The equations  $E_{U^\mu}F(x, U, \partial U, \dots, \partial^s U) \equiv 0$ ,  $\mu = 1, \dots, m$  hold for arbitrary  $U(x)$  if and only if  $F(x, U, \partial U, \dots, \partial^s U) \equiv D_i\Psi^i(x, U, \partial U, \dots, \partial^{s-1}U)$  for some functions  $\Psi^i(x, U, \partial U, \dots, \partial^{s-1}U)$ ,  $i = 1, \dots, n$ .*

From Theorem 2.1, the proof of the following theorem that connects local multipliers and local conservation laws is immediate.

**Theorem 2.2.** *A set of non-singular local multipliers  $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$  yields a divergence expression for a DE system  $\mathbf{R}\{x; u\}$  (1) if and only if the set of equations*

$$E_{U^\mu}(\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)R^\sigma(x, U, \partial U, \dots, \partial^k U)) \equiv 0, \quad \mu = 1, \dots, m, \quad (5)$$

*holds for arbitrary functions  $U(x)$ .*

The set of equations (5) yields the set of *linear determining equations* to find *all* sets of local conservation law multipliers of a given DE system  $\mathbf{R}\{x; u\}$  (1) by letting  $l = 1, 2, \dots$  in (5). Since the equations (1) hold for arbitrary  $U(x)$ , it follows that they also hold for *each* derivative of  $U(x)$  replaced by an arbitrary function. In particular, since derivatives of  $U(x)$  of orders higher than  $l$  can be replaced by arbitrary functions, it follows that the linear PDE system (1) splits into an over-determined linear system of determining equations whose solutions are the sets of local multipliers  $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$  of the DE system  $\mathbf{R}\{x; u\}$  (1).

One can show the following [11]: Suppose each DE of a given  $k$ th order DE system  $\mathbf{R}\{x; u\}$  (1) can be written in a solved form

$$R^\sigma[u] = u_{i_{1\sigma}\dots i_{s\sigma}}^{j_\sigma} - G^\sigma(x, u, \partial u, \dots, \partial^k u) = 0, \quad \sigma = 1, \dots, N, \quad (6)$$

where  $1 \leq j_\sigma \leq m$  and  $1 \leq i_{1\sigma}, \dots, i_{s\sigma} \leq n$  for each  $\sigma = 1, \dots, N$ ;  $\{u_{i_{1\sigma}\dots i_{s\sigma}}^{j_\sigma}\}$  is a set of  $N$  linearly independent  $s$ th order leading (partial) derivatives, with the property that none of them or their differential consequences appears in  $\{G^\sigma[u]\}_{\sigma=1}^N$ . Then, to within equivalence, *all* local conservation laws of the DE system  $\mathbf{R}\{x; u\}$  (1) arise from sets of local multipliers that are solutions of the determining equations (5). [It should be noted that the assumption that a given DE system  $\mathbf{R}\{x; u\}$  (1) can be written in a solved form (6) is the same assumption that is required when one is finding the local symmetries of  $\mathbf{R}\{x; u\}$  (1).]

**Remark 2.1.** In the situation when a given DE system  $\mathbf{R}\{x; u\}$  (1) cannot be written in a solved form (6), the multiplier approach still can be used to see local conservation laws of (1). However, here it is possible that some local conservation laws are missed since the corresponding divergence expressions may not satisfy (3), since they could involve differential consequences of  $\mathbf{R}\{x; u\}$  (1).

Following from the above, a systematic procedure for the construction of local conservation laws of a given DE system  $\mathbf{R}\{x; u\}$  (1), referred to as the *direct method*, is now outlined.

- For a given  $k$ th order DE system  $\mathbf{R}\{x; u\}$  (1), seek sets of multipliers of the form  $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$  to some specified order  $l$ . Choose the dependence of multipliers on their arguments so that singular multipliers do not arise. [In particular, if the given DE system is written in a solved form (6) and is non-degenerate, the multipliers can be assumed to have no dependence on the leading derivatives  $\{u_{i_{1\sigma}\dots i_{s\sigma}}^{j_\sigma}\}$  and their differential consequences.]
- Solve the set of determining equations (5) for arbitrary  $U(x)$  to find all such sets of multipliers.
- Find the corresponding fluxes  $\Phi^i(x, U, \partial U, \dots, \partial^r U)$  satisfying the identity
 
$$\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U) R^\sigma(x, U, \partial U, \dots, \partial^k U) \equiv D_i \Phi^i(x, U, \partial U, \dots, \partial^r U).$$
- Each set of multipliers and resulting fluxes yields a local conservation law holding for all solutions  $u(x)$  of the given DE system  $\mathbf{R}\{x; u\}$  (1).

## 2.1 Examples

The direct method to obtain local conservation laws is now illustrated through two examples.

### 2.1.1 A nonlinear telegraph system

As a first example, consider a nonlinear telegraph system ( $u^1 = u$ ,  $u^2 = v$ ) given by

$$R^1[u, v] = v_t - (u^2 + 1)u_x - u = 0, \quad R^2[u, v] = u_t - v_x = 0. \quad (7)$$

This is a first order PDE system with leading derivatives  $v_t$  and  $u_t$ .

We seek all local conservation law multipliers of the form

$$\Lambda_1 = \xi(x, t, U, V), \quad \Lambda_2 = \phi(x, t, U, V) \quad (8)$$

of the PDE system (7). In terms of Euler operators

$$E_U = \frac{\partial}{\partial U} - D_x \frac{\partial}{\partial U_x} - D_t \frac{\partial}{\partial U_t}, \quad E_V = \frac{\partial}{\partial V} - D_x \frac{\partial}{\partial V_x} - D_t \frac{\partial}{\partial V_t},$$

the determining equations (5) for the multipliers (8) become

$$\begin{aligned} E_U[\xi(x, t, U, V)(V_t - (U^2 + 1)U_x - U) + \phi(x, t, U, V)(U_t - V_x)] &\equiv 0, \\ E_V[\xi(x, t, U, V)(V_t - (U^2 + 1)U_x - U) + \phi(x, t, U, V)(U_t - V_x)] &\equiv 0, \end{aligned} \quad (9)$$

where  $U(x, t)$  and  $V(x, t)$  are arbitrary differentiable functions. Equations (9) split with respect to  $U_t, V_t, U_x, V_x$  to yield the over-determined linear PDE system given by

$$\begin{aligned} \phi_V - \xi_U &= 0, \quad \phi_U - (U^2 + 1)\xi_V = 0, \\ \phi_x - \xi_t - U\xi_V &= 0, \quad (U^2 + 1)\xi_x - \phi_t - U\xi_U - \xi = 0. \end{aligned} \quad (10)$$

The solutions of (10) are the five sets of local conservation multipliers given by

$$\begin{aligned} (\xi_1, \phi_1) &= (0, 1), \quad (\xi_2, \phi_2) = (t, x - \frac{1}{2}t^2), \\ (\xi_3, \phi_3) &= (1, -t), \quad (\xi_4, \phi_4) = (e^{x+\frac{1}{2}U^2+V}, Ue^{x+\frac{1}{2}U^2+V}), \\ (\xi_5, \phi_5) &= (e^{x+\frac{1}{2}U^2-V}, -Ue^{x+\frac{1}{2}U^2-V}). \end{aligned}$$

Each set  $(\xi, \phi)$  determines a nontrivial local conservation law  $D_t\Psi(x, t, u, v) + D_x\Phi(x, t, u, v) = 0$  with the characteristic form

$$\begin{aligned} D_t\Psi(x, t, U, V) + D_x\Phi(x, t, U, V) \\ \equiv \xi(x, t, U, V)R^1[U, V] + \phi(x, t, U, V)R^2[U, V]. \end{aligned} \quad (11)$$

In particular, after equating like derivative terms of (11), one has the relations

$$\Psi_U = \phi, \quad \Psi_V = \xi, \quad \Phi_U = -(U^2 + 1)\xi, \quad \Phi_V = -\phi, \quad \Psi_t + \Phi_x = -U\xi. \quad (12)$$

For each set of local multipliers, it is straightforward to integrate equations (12) to obtain the following five linearly independent local conservation laws of the PDE system (7):

$$\begin{aligned} D_t u + D_x[-v] &= 0, \\ D_t[(x - \frac{1}{2}t^2)u + tv] + D_x[(\frac{1}{2}t^2 - x)v - t(\frac{1}{3}u^3 + u)] &= 0, \\ D_t[v - tu] + D_x[tv - (\frac{1}{3}u^3 + u)] &= 0, \\ D_t[e^{x+\frac{1}{2}u^2+v}] + D_x[-ue^{x+\frac{1}{2}u^2+v}] &= 0, \\ D_t[e^{x+\frac{1}{2}u^2-v}] + D_x[ue^{x+\frac{1}{2}u^2-v}] &= 0. \end{aligned}$$

### 2.1.2 Korteweg-de Vries equation

As a second example, consider the KdV equation

$$R[u] = u_t + uu_x + u_{xxx} = 0. \quad (13)$$

Since PDE (13) can be directly expressed in the solved form  $u_t = g[u] = -(uu_x + u_{xxx})$ , without loss of generality, it follows that local multipliers yielding local conservation laws of PDE (13) are of the form  $\Lambda = \Lambda(t, x, U, \partial_x U, \dots, \partial_x^l U)$ ,  $l = 1, 2, \dots$ , i.e., multipliers can be assumed to depend on at most on  $x$ -derivatives of  $U$ . This follows from the observation that through PDE (13), all  $t$ -derivatives of  $u$  appearing in the fluxes of any local conservation law  $D_t \Psi[u] + D_x \Phi[u] = 0$  of PDE (13) can be expressed in terms of  $x$ -derivatives of  $u$ . It is then easy to show [3] that the resulting multipliers for the fluxes  $\Psi(t, x, U, \partial_x U, \dots, \partial_x^r U)$  and  $\Phi(t, x, U, \partial_x U, \dots, \partial_x^r U)$  must have no dependence on  $U_t$  and its derivatives. Consequently,  $\Lambda(t, x, U, \partial_x U, \dots, \partial_x^l U)$  is a local conservation law multiplier of the PDE (13) if and only if

$$\begin{aligned} E_U(\Lambda(t, x, U, \partial_x U, \dots, \partial_x^l U)(U_t + UU_x + U_{xxx})) &\equiv \\ -D_t \Lambda - U D_x \Lambda - D_x^3 \Lambda + (U_t + UU_x + U_{xxx}) \Lambda_U & \\ -D_x((U_t + UU_x + U_{xxx}) \Lambda_{\partial_x U}) & \\ + \dots + (-1)^l D_x^l((U_t + UU_x + U_{xxx}) \Lambda_{\partial_x^l U}) &\equiv 0 \end{aligned} \quad (14)$$

holds for an *arbitrary*  $U(x, t)$  where here the Euler operator

$$E_U = \frac{\partial}{\partial u} - (D_t \frac{\partial}{\partial U_t} + D_x \frac{\partial}{\partial U_x}) + D_x^2 \frac{\partial}{\partial U_{xx}} + \dots$$

truncates after  $\max(3, l)$   $x$ -derivatives of  $U$ . Note that the linear determining equation (14) is of the form

$$\alpha_1[U] + \alpha_2[U]U_t + \alpha_3[U]\partial_x U_t + \dots + \alpha_{l+2}[U]\partial_x^l U_t \equiv 0 \quad (15)$$



where each  $\alpha_i[U]$  depends at most on  $t, x, U$  and  $x$ -derivatives of  $U$ . Since  $U(x, t)$  is an arbitrary function, in equation (15) each of  $U_t, \partial_x U_t, \dots, \partial_x^l U_t$  can be treated as independent variables, and hence  $\alpha_i[U] = 0, i = 1, \dots, l + 2$ . Furthermore, there is a further splitting of these  $l + 2$  determining equations with respect to each  $x$ -derivative of  $U$ .

Now suppose  $\Lambda = \Lambda(t, x, U)$ . Then from equations (14) and (15), it follows that

$$\begin{aligned} & (\Lambda_t + U\Lambda_x + \Lambda_{xxx}) + 3\Lambda_{xxU}U_x + 3\Lambda_{xUU}U_x^2 + \Lambda_{UUU}U_x^3 \\ & + 3\Lambda_{xU}U_{xx} + 3\Lambda_{UU}U_xU_{xx} \equiv 0. \end{aligned} \quad (16)$$

Equation (16) is a polynomial identity in the variables  $U_x, U_{xx}$ . Hence equation (16) splits into the three equations (the other three equations are differential consequences)

$$\Lambda_t + U\Lambda_x + \Lambda_{xxx} = 0, \quad \Lambda_{xU} = 0, \quad \Lambda_{UU} = 0,$$

whose solution yields the three local conservation law multipliers

$$\Lambda_1 = 1, \quad \Lambda_2 = U, \quad \Lambda_3 = tU - x.$$

It is easy to check that these three multipliers respectively yield the divergence expressions

$$\begin{aligned} & U_t + UU_x + U_{xxx} \equiv D_t U + D_x \left( \frac{1}{2} U^2 + U_{xx} \right), \\ & U(U_t + UU_x + U_{xxx}) \equiv D_t \left( \frac{1}{2} U^2 \right) + D_x \left( \frac{1}{3} U^3 + UU_{xx} - \frac{1}{2} U_x^2 \right), \\ & (tU - x)(U_t + UU_x + U_{xxx}) \equiv D_t \left( \frac{1}{2} tU^2 - xU \right) \\ & + D_x \left( -\frac{1}{2} xU^2 + tUU_{xx} - \frac{1}{2} tU_x^2 - xU_{xx} + U_x \right), \end{aligned}$$

and consequently, one obtains the local conservation laws

$$\begin{aligned} & D_t u + D_x \left( \frac{1}{2} u^2 + u_{xx} \right) = 0, \\ & D_t \left( \frac{1}{2} u^2 \right) + D_x \left( \frac{1}{3} u^3 + uu_{xx} - \frac{1}{2} u_x^2 \right) = 0, \\ & D_t \left( \frac{1}{2} tu^2 - xu \right) + D_x \left( -\frac{1}{2} xu^2 + tuu_{xx} - \frac{1}{2} tu_x^2 - xu_{xx} + u_x \right) = 0, \end{aligned}$$

of the KdV equation (13).

From equations (14) and (15), it is easy to see that PDE (13) has no additional multipliers of the form  $\Lambda = \Lambda(t, x, U, U_x)$  with an essential dependence on  $U_x$ . Moreover, one can show that there is only one additional local multiplier of the form  $\Lambda = \Lambda(t, x, U, U_x, U_{xx})$ , given by

$$\Lambda_4 = U_{xx} + \frac{1}{2} U^2.$$

Furthermore, one can show that in terms of the recursion operator

$$R^*[U] = D_x^2 + \frac{1}{3} U + \frac{1}{3} D_x^{-1} \circ U \circ D_x,$$

the KdV equation (13) has an infinite sequence of local conservation law multipliers given by

$$\Lambda_{2n} = (\mathbf{R}^*[U])^n U, \quad n = 1, 2, \dots,$$

with the first two multipliers in this sequence exhibited above.

## 2.2 Linearizing operators and adjoint equations

Consider a given DE system  $\mathbf{R}\{x; u\}$  (1). The *linearizing operator*  $L[U]$  associated with the DE system  $\mathbf{R}\{x; u\}$  (1) is given by

$$L_\rho^\sigma[U]V^\rho = \left[ \frac{\partial R^\sigma[U]}{\partial U^\rho} + \frac{\partial R^\sigma[U]}{\partial U_i^\rho} D_i + \dots + \frac{\partial R^\sigma[U]}{\partial U_{i_1 \dots i_k}^\rho} D_{i_1} \dots D_{i_k} \right] V^\rho, \quad (17)$$

$$\sigma = 1, \dots, N,$$

in terms of an *arbitrary* function  $V(x) = (V^1(x), \dots, V^m(x))$ . The *adjoint operator*  $L^*[U]$  associated with the DE system  $\mathbf{R}\{x; u\}$  (1) is given by

$$L_\rho^{*\sigma}[U]W_\sigma = \frac{\partial R^\sigma[U]}{\partial U^\rho} W_\sigma - D_i \left( \frac{\partial R^\sigma[U]}{\partial U_i^\rho} W_\sigma \right) + \dots$$

$$+ (-1)^k D_{i_1} \dots D_{i_k} \left( \frac{\partial R^\sigma[U]}{\partial U_{i_1 \dots i_k}^\rho} W_\sigma \right), \quad \rho = 1, \dots, m, \quad (18)$$

in terms of an *arbitrary* function  $W(x) = (W_1(x), \dots, W_N(x))$ .

In particular, one can show that the linearizing and adjoint operators, defined respectively through (17) and (18), satisfy the divergence relation

$$W_\sigma L_\rho^\sigma[U]V^\rho - V^\rho L_\rho^{*\sigma}[U]W_\sigma \equiv D_i \Psi^i[U]$$

with

$$D_i \Psi^i[U] = \sum_{q=1}^k \sum_{i_1 \dots i_q} D_{i_m} \left[ (-1)^{m-1} (D_{i_{m+1}} \dots D_{i_q} V^\rho) \times \right.$$

$$\left. \times D_{i_1} \dots D_{i_{m-1}} \left( W_\sigma \frac{\partial R^\sigma[U]}{\partial U_{i_1 \dots i_q}^\rho} \right) \right],$$

where the second sum is taken over all ordered sets of indices  $1 \leq i_1 \leq \dots \leq i_m \leq \dots \leq i_q \leq n$  of independent variables  $x = (x^1, \dots, x^n)$ .

Now let  $W_\sigma = \Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)$ ,  $\sigma = 1, \dots, N$ . By direct calculation, in terms of the Euler operators defined by (4), one can show that

$$E_{U^\rho}(\Lambda_\sigma[U]R^\sigma[U]) \equiv L_\rho^{*\sigma}[U]\Lambda_\sigma[U] + F_\rho(R[U]) \quad (19)$$

with

$$\begin{aligned}
F_\rho(R[U]) &= \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} R^\sigma[U] - D_i \left( \frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho} R^\sigma[U] \right) + \dots \\
&+ (-1)^l D_{i_1} \dots D_{i_l} \left( \frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1 \dots i_l}^\rho} R^\sigma[U] \right), \quad \rho = 1, \dots, m.
\end{aligned} \tag{20}$$

From expression (19), it immediately follows that  $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$  yields a set of local conservation law multipliers of the DE system  $\mathbf{R}\{x; u\}$  (1) if and only if the right hand side of (19) vanishes for arbitrary  $U(x)$ . Now suppose each multiplier is nonsingular for each solution  $U(x) = u(x)$  of the DE system (1). Since then the expression (20) vanishes for each solution  $U(x) = u(x)$  of DE system  $\mathbf{R}\{x; u\}$  (1), it follows that every set of nonsingular multipliers  $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$  of  $\mathbf{R}\{x; u\}$  is a solution of its adjoint linearizing DE system when  $U(x) = u(x)$  is a solution of the DE system  $\mathbf{R}\{x; u\}$ , i.e.,

$$L_\rho^{*\sigma}[u] \Lambda_\sigma[u] = 0, \quad \rho = 1, \dots, m. \tag{21}$$

In particular, the following two results have been proved.

**Theorem 2.3.** *For a given DE system  $\mathbf{R}\{x; u\}$  (1), each set of local conservation law multipliers  $\{\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$  satisfies the identity*

$$\begin{aligned}
L_\rho^{*\sigma}[U] \Lambda_\sigma[U] + \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} R^\sigma[U] - D_i \left( \frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho} R^\sigma[U] \right) \\
+ \dots + (-1)^l D_{i_1} \dots D_{i_l} \left( \frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1 \dots i_l}^\rho} R^\sigma[U] \right) \equiv 0, \quad \rho = 1, \dots, m,
\end{aligned} \tag{22}$$

holding for arbitrary functions  $U(x) = (U^1(x), \dots, U^m(x))$  where the components  $\{L_\rho^{*\sigma}[U]\}$  of the adjoint operator of the linearizing operator (Fréchet derivative) for the DE system (1) are given by expressions (18).

**Corollary 2.1.** *For any solution  $U(x) = u(x) = (u^1(x), \dots, u^m(x))$  of a given DE system  $\mathbf{R}\{x; u\}$  (1), each set of local conservation law multipliers  $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$  satisfies the adjoint linearizing system (21), where  $\{L_\rho^{*\sigma}[U]\}$  is given by the components of the adjoint operator (18).*

The identity (22) provides the explicit general form of the multiplier determining system (5) in Theorem 2.2. In general, the adjoint system (21) is strictly a subset of system (5) after one takes into account the splitting of (22) with respect to a set of leading derivatives for  $R^\sigma[U]$ ,  $\sigma = 1, \dots, N$ .

### 2.3 Determination of fluxes of conservation laws from multipliers

There are several ways of finding the fluxes of local conservation laws from a known set of multipliers.

A first method is a direct method that has been illustrated through the nonlinear telegraph system considered in Section 2.1.1 where one converts (3) directly into the set of determining equations to be solved for the fluxes  $\Phi^i[U]$ . This method is easy to implement for simple types of conservation laws.

A second method is another direct method that has been illustrated through the KdV equation considered in Section 2.1.2 where one simply manipulates (3) to find the fluxes  $\Phi^i[U]$ .

A third method [1–3] is now presented that allows one to find the fluxes in the case of complicated forms of multipliers and/or DE systems through an integral (homotopy) formula:

For each multiplier  $\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)$ , one introduces the corresponding linearization operator

$$(\mathbf{L}_\Lambda)_{\sigma\rho}[U]\tilde{V}^\rho = \left[ \frac{\partial\Lambda_\sigma[U]}{\partial U^\rho} + \frac{\partial\Lambda_\sigma[U]}{\partial U_i^\rho} D_i + \dots + \frac{\partial\Lambda_\sigma[U]}{\partial U_{i_1\dots i_l}^\rho} D_{i_1} \dots D_{i_l} \right] \tilde{V}^\rho, \quad (23)$$

$$\sigma = 1, \dots, N,$$

and its adjoint

$$(\mathbf{L}_\Lambda^*)_{\sigma\rho}[U]\tilde{W}^\sigma = \frac{\partial\Lambda_\sigma[U]}{\partial U^\rho} \tilde{W}^\sigma - D_i \left( \frac{\partial\Lambda_\sigma[U]}{\partial U_i^\rho} \tilde{W}^\sigma \right) \\ + \dots + (-1)^k D_{i_1} \dots D_{i_l} \left( \frac{\partial\Lambda_\sigma[U]}{\partial U_{i_1\dots i_l}^\rho} \tilde{W}^\sigma \right), \quad \rho = 1, \dots, m, \quad (24)$$

acting respectively on arbitrary functions  $\tilde{V}(x) = (\tilde{V}^1(x), \dots, \tilde{V}^m(x))$  and  $\tilde{W}(x) = (\tilde{W}^1(x), \dots, \tilde{W}^N(x))$ .

It is straightforward to show that the operators defined by (17), (18), (23), and (24) satisfy the following divergence identities:

$$W_\sigma L_\rho^\sigma[U] V^\rho - V^\rho L_\rho^* \sigma[U] W_\sigma \equiv D_i S^i[V, W; R[U]], \quad (25)$$

$$\tilde{W}^\sigma (\mathbf{L}_\Lambda)_{\sigma\rho}[U] \tilde{V}^\rho - \tilde{V}^\rho (\mathbf{L}_\Lambda^*)_{\sigma\rho}[U] \tilde{W}^\sigma \equiv D_i \tilde{S}^i[\tilde{V}, \tilde{W}; \Lambda[U]], \quad (26)$$

with  $S^i[V, W; R[U]]$  and  $\tilde{S}^i[\tilde{V}, \tilde{W}; \Lambda[U]]$  defined by corresponding terms in the expressions

$$D_i S^i[V, W; R[U]] = \sum_{q=1}^k \sum_{i_1\dots i_q} D_{i_m} \left[ (-1)^{m-1} (D_{i_{m+1}} \dots D_{i_q} V^\rho) \times \right. \\ \left. \times D_{i_1} \dots D_{i_{m-1}} \left( W_\sigma \frac{\partial R^\sigma[U]}{\partial U_{i_1\dots i_q}^\rho} \right) \right], \quad (27)$$

$$D_i \tilde{S}^i[\tilde{V}, \tilde{W}; \Lambda[U]] = \sum_{q=1}^l \sum_{i_1\dots i_q} D_{i_m} \left[ (-1)^{m-1} (D_{i_{m+1}} \dots D_{i_q} \tilde{V}^\rho) \times \right. \\ \left. \times D_{i_1} \dots D_{i_{m-1}} \left( \tilde{W}^\sigma \frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1\dots i_q}^\rho} \right) \right]. \quad (28)$$

In equations (27) and (28),  $k$  is the order of the given DE system (1),  $l$  is the maximal order of the derivatives appearing in the multipliers, and the second sums are taken over all ordered sets of indices  $1 \leq i_1 \leq \dots \leq i_m \leq \dots \leq i_q \leq n$  of independent variables  $x = (x^1, \dots, x^n)$ .

Let  $U_{(\lambda)} = U(x) + (\lambda - 1)V(x)$ , where  $U(x) = (U^1(x), \dots, U^m(x))$  and  $V(x) = (V^1(x), \dots, V^m(x))$  are arbitrary functions, and  $\lambda$  is a scalar parameter. Replacing  $U$  by  $U_{(\lambda)}$  in the conservation law identity (3), one obtains

$$\frac{\partial}{\partial \lambda} (\Lambda_\sigma[U_{(\lambda)}] R^\sigma[U_{(\lambda)}]) \equiv \frac{\partial}{\partial \lambda} D_i \Phi^i[U_{(\lambda)}] = D_i \left( \frac{\partial}{\partial \lambda} \Phi^i[U_{(\lambda)}] \right). \quad (29)$$

The left-hand side of (29) can then be expressed in terms of the linearizing operators (17) and (23) as follows:

$$\frac{\partial}{\partial \lambda} (\Lambda_\sigma[U_{(\lambda)}] R^\sigma[U_{(\lambda)}]) = \Lambda_\sigma[U_{(\lambda)}] L_\rho^\sigma[U_{(\lambda)}] V^\rho + R^\sigma[U_{(\lambda)}] (L_\Lambda)_{\sigma\rho}[U_{(\lambda)}] V^\rho.$$

From (25) and (26) with  $W_\sigma = \Lambda_\sigma[U_{(\lambda)}]$  and  $\tilde{W}^\sigma = R^\sigma[U_{(\lambda)}]$ , respectively, one obtains

$$\begin{aligned} \frac{\partial}{\partial \lambda} (\Lambda_\sigma[U_{(\lambda)}] R^\sigma[U_{(\lambda)}]) &= V^\rho L_\rho^* \Lambda_\sigma[U_{(\lambda)}] + D_i S^i[V, \Lambda[U_{(\lambda)}]; R[U_{(\lambda)}]] \\ &+ V^\rho (L_\Lambda^*)_{\sigma\rho}[U_{(\lambda)}] R^\sigma[U_{(\lambda)}] + D_i \tilde{S}^i[V, R[U_{(\lambda)}]; \Lambda[U_{(\lambda)}]] \\ &= D_i \left( S^i[V, \Lambda[U_{(\lambda)}]; R[U_{(\lambda)}]] + \tilde{S}^i[V, R[U_{(\lambda)}]; \Lambda[U_{(\lambda)}]] \right), \end{aligned} \quad (30)$$

where the last equality follows from the identity (22) holding for local conservation law multipliers in Theorem 2.3.

Comparing (29) and (30), one finds that

$$D_i \left( \frac{\partial}{\partial \lambda} \Phi^i[U_{(\lambda)}] \right) = D_i \left( S^i[V, \Lambda[U_{(\lambda)}]; R[U_{(\lambda)}]] + \tilde{S}^i[V, R[U_{(\lambda)}]; \Lambda[U_{(\lambda)}]] \right),$$

leading to

$$\frac{\partial}{\partial \lambda} \Phi^i[U_{(\lambda)}] = S^i[V, \Lambda[U_{(\lambda)}]; R[U_{(\lambda)}]] + \tilde{S}^i[V, R[U_{(\lambda)}]; \Lambda[U_{(\lambda)}]], \quad (31)$$

up to fluxes of a trivial conservation law. Now let  $V(x) = U(x) - \tilde{U}(x)$ , for an arbitrary function  $\tilde{U}(x) = (\tilde{U}^1(x), \dots, \tilde{U}^m(x))$ . Then  $U_{(\lambda)} = \lambda U(x) + (1 - \lambda)\tilde{U}(x)$ . Integrating (31) with respect to  $\lambda$  from 0 to 1, one finds that

$$\begin{aligned} \Phi^i[U] &= \Phi^i[\tilde{U}] + \int_0^1 (S^i[U - \tilde{U}, \Lambda[\lambda U + (1 - \lambda)\tilde{U}]; R[\lambda U + (1 - \lambda)\tilde{U}]] \\ &+ \tilde{S}^i[U - \tilde{U}, R[\lambda U + (1 - \lambda)\tilde{U}]; \Lambda[\lambda U + (1 - \lambda)\tilde{U}]] d\lambda, \end{aligned} \quad (32)$$

$$i = 1, \dots, n.$$

In summary, the following theorem has been proven.

**Theorem 2.4.** For a set of local conservation law multipliers  $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$  of a DE system  $\mathbf{R}\{x; u\}$  (1), the corresponding fluxes are given by the integral formula (32).

In (32),  $\tilde{U}(x)$  is an arbitrary function of  $x$ , chosen so that the integral converges. Different choices of  $\tilde{U}(x)$  yield fluxes of equivalent conservation laws, i.e., conservation laws that differ by trivial divergences. One commonly chooses  $\tilde{U}(x) = 0$  (provided that the integral (32) converges). Once  $\tilde{U}(x)$  has been chosen, the corresponding fluxes  $\{\Phi^i[\tilde{U}]\}_{i=1}^N$  can be found by direct integration through the divergence relation  $D_i\Phi^i[\tilde{U}] = \Lambda_\sigma[\tilde{U}]R^\sigma[\tilde{U}] = F(x)$ . For example, one may choose  $\Phi^1[\tilde{U}] = \int F(x) dx^1$ ,  $\Phi^2[\tilde{U}] = \dots = \Phi^n[\tilde{U}] = 0$ .

Finally, a fourth method [12] replaces the integral formula (32) by a simpler algebraic formula that applies to DE systems  $\mathbf{R}\{x; u\}$  that have scaling symmetries.

## 2.4 Self-adjoint DE systems

An especially interesting situation arises when the linearizing operator (Fréchet derivative)  $L[U]$  of a given DE system (1) is self-adjoint.

**Definition 2.4.** Let  $L[U]$ , with its components  $L_\rho^\sigma[U]$  given by (17), be the linearizing operator associated with a DE system (1). The adjoint operator of  $L[U]$  is  $L^*[U]$ , with its components  $L_\rho^{*\sigma}[U]$  given by (18).  $L[U]$  is a *self-adjoint* operator if and only if  $L[U] \equiv L^*[U]$ , i.e.,  $L_\rho^\sigma[U] \equiv L_\rho^{*\sigma}[U]$ ,  $\sigma, \rho = 1, \dots, m$ .

It is straightforward to see that if a DE system, as written, has a self-adjoint linearizing operator, then

- the number of dependent variables appearing in the system must equal the number of equations appearing in the system, i.e.,  $N = m$ ;
- if the given DE system is a scalar equation, the highest-order derivative appearing in it must be of even order.

The converse of this statement is false. For example, consider the linear heat equation  $u_t - u_{xx} = 0$ . The linearizing operator of this PDE is obviously given by  $L = D_t - D_x^2$ , with adjoint operator  $L^* = -D_t - D_x^2 \neq L$ .

Most importantly, one can show that a given DE system, as written, has a variational formulation if and only if its associated linearizing operator is self-adjoint [8–10].

If the linearizing operator associated with a given DE system is self-adjoint, then each set of local conservation law multipliers yields a local symmetry of the given DE system. In particular, one has the following theorem.

**Theorem 2.5.** Consider a given DE system  $\mathbf{R}\{x; u\}$  (1) with  $N = m$ . Suppose its associated linearizing operator  $L[U]$ , with components (17), is self-adjoint. Suppose  $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^m$  is a set of local conservation law multipliers of the DE system (1). Let  $\eta^\sigma(x, u, \partial u, \dots, \partial^l u) = \Lambda_\sigma(x, u, \partial u, \dots, \partial^l u)$ ,

$\sigma = 1, \dots, m$ , where  $U(x) = u(x)$  is any solution of the DE system  $\mathbf{R}\{x; u\}$  (1). Then

$$\eta^\sigma(x, u, \partial u, \dots, \partial^l u) \frac{\partial}{\partial u^\sigma} \quad (33)$$

is a local symmetry of the DE system  $\mathbf{R}\{x; u\}$  (1).

**Proof.** From equations (21) with  $L[U] = L^*[U]$ , it follows that in terms of the components (17) of the associated linearizing operator  $L[U]$ , one has

$$L_\rho^\sigma[u] \Lambda_\sigma(x, u, \partial u, \dots, \partial^l u) = 0, \quad \rho = 1, \dots, m, \quad (34)$$

where  $u = \Theta(x)$  is any solution of the DE system  $\mathbf{R}\{x; u\}$  (1). But the set of equations (34) is the set of determining equations for a local symmetry

$$\Lambda_\sigma(x, u, \partial u, \dots, \partial^l u) \frac{\partial}{\partial u^\sigma}$$

of the DE system  $\mathbf{R}\{x; u\}$  (1). Hence, it follows that (33) is a local symmetry of the DE system  $\mathbf{R}\{x; u\}$  (1). ■

The converse of Theorem 2.5 is false. In particular, suppose

$$\eta^\sigma(x, u, \partial u, \dots, \partial^l u) \frac{\partial}{\partial u^\sigma}$$

is a local symmetry of a given DE system  $\mathbf{R}\{x; u\}$  (1) with a self-adjoint linearizing operator  $L[U]$ . Let  $\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U) = \eta^\sigma(x, U, \partial U, \dots, \partial^l U)$ ,  $\sigma = 1, \dots, m$ , where  $U(x) = (U^1(x), \dots, U^m(x))$  is an arbitrary function. Then it does not necessarily follow that  $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^m$  is a set of local conservation law multipliers of the DE system (1). This can be seen as follows: in the self-adjoint case, the set of local symmetry determining equations is a subset of the set of local multiplier determining equations. Here *each* local symmetry yields a set of local conservation law multipliers if and only *each* solution of the set of local symmetry determining equations also solves the remaining set of local multiplier determining equations.

### 3 Noether's theorem

In 1918, Noether [5] presented her celebrated procedure (Noether's theorem) to find local conservation laws for systems of DEs that admit a variational principle. When a given DE system admits a variational principle, then the extremals of an action functional yield the given DE system (the *Euler–Lagrange equations*). In this case, Noether showed that if one has a point symmetry of the action functional (action integral), then one obtains the fluxes of a local conservation law through an explicit formula that involves the infinitesimals of the point symmetry and the Lagrangian (Lagrangian density) of the action functional.

We now present Noether's theorem and its generalizations due to Bessel-Hagen [6] and Boyer [7].

### 3.1 Euler–Lagrange equations

Consider a functional  $J[U]$  in terms of  $n$  independent variables  $x = (x^1, \dots, x^n)$  and  $m$  arbitrary functions  $U = (U^1(x), \dots, U^m(x))$  and their derivatives to order  $k$ , defined on a domain  $\Omega$ ,

$$J[U] = \int_{\Omega} L[U] dx = \int_{\Omega} L[(x, U, \partial U, \dots, \partial^k U)] dx. \quad (35)$$

The function  $L[U] = L[(x, U, \partial U, \dots, \partial^k U)]$  is called a *Lagrangian* and the functional  $J[U]$  is called an *action integral*. Consider an infinitesimal change of  $U$  given by  $U(x) \rightarrow U(x) + \varepsilon v(x)$  where  $v(x)$  is any function such that  $v(x)$  and its derivatives to order  $k - 1$  vanish on the boundary  $\partial\Omega$  of the domain  $\Omega$ . The corresponding change (variation) in the Lagrangian  $L[U]$  is given by

$$\begin{aligned} \delta L &= L[(x, U + \varepsilon v, \partial U + \varepsilon \partial v, \dots, \partial^k U + \varepsilon \partial^k v)] - L[(x, U, \partial U, \dots, \partial^k U)] \\ &= \varepsilon \left( \frac{\partial L[U]}{\partial U^\sigma} v^\sigma + \frac{\partial L[U]}{\partial U_j^\sigma} v_j^\sigma + \dots + \frac{\partial L[U]}{\partial U_{j_1 \dots j_k}^\sigma} v_{j_1 \dots j_k}^\sigma \right) + O(\varepsilon^2). \end{aligned}$$

Then after repeatedly using integration by parts, one can show that

$$\delta L = \varepsilon (v^\sigma E_{U^\sigma}(L[U]) + D_l W^l[U, v]) + O(\varepsilon^2), \quad (36)$$

where  $E_{U^\sigma}$  is the Euler operator with respect to  $U^\sigma$  and

$$\begin{aligned} W^l[U, v] &= v^\sigma \left( \frac{\partial L[U]}{\partial U_l^\sigma} + \dots + (-1)^{k-1} D_{j_1} \dots D_{j_{k-1}} \frac{\partial L[U]}{\partial U_{l j_1 \dots j_{k-1}}^\sigma} \right) \\ &+ v_{j_1}^\sigma \left( \frac{\partial L[U]}{\partial U_{j_1 l}^\sigma} + \dots + (-1)^{k-2} D_{j_2} \dots D_{j_{k-1}} \frac{\partial L[U]}{\partial U_{j_1 l j_2 \dots j_{k-1}}^\sigma} \right) \\ &+ \dots + v_{j_1 \dots j_{k-1}}^\sigma \frac{\partial L[U]}{\partial U_{j_1 j_2 \dots j_{k-1} l}^\sigma}. \end{aligned} \quad (37)$$

The corresponding variation in the action integral  $J[U]$  is given by

$$\begin{aligned} \delta J &= J[U + \varepsilon v] - J[U] = \int_{\Omega} \delta L dx \\ &= \varepsilon \int_{\Omega} (v^\sigma E_{U^\sigma}(L[U]) + D_l W^l[U, v]) dx + O(\varepsilon^2) \\ &= \varepsilon \left( \int_{\Omega} v^\sigma E_{U^\sigma}(L[U]) dx + \int_{\partial\Omega} W^l[U, v] n^l dS \right) + O(\varepsilon^2) \end{aligned} \quad (38)$$

where  $\int_{\partial\Omega}$  represents the surface integral over the boundary  $\partial\Omega$  of the domain  $\Omega$  with  $n = (n^1, \dots, n^n)$  being the unit outward normal vector to  $\partial\Omega$ . From (37), it is evident that each  $W^l[U, v]$  vanishes on  $\partial\Omega$ , and hence  $\int_{\partial\Omega} W^l[U, v] n^l dS = 0$ .

Hence if  $U = u(x)$  extremizes the action integral  $J[U]$ , then the  $O(\varepsilon)$  term of  $\delta J$  must vanish so that  $\int_{\Omega} v^\sigma E_{u^\sigma}(L[u]) dx = 0$  for an *arbitrary*  $v(x)$  defined on the



domain  $\Omega$ . Thus if  $U = u(x)$  extremizes the action integral (35), then  $u(x)$  must satisfy the *Euler–Lagrange equations*

$$E_{u^\sigma}(L[u]) = \frac{\partial L[u]}{\partial u^\sigma} + \cdots + (-1)^k D_{j_1} \cdots D_{j_k} \frac{\partial L[u]}{\partial u_{j_1 \cdots j_k}^\sigma} = 0, \quad \sigma = 1, \dots, m. \quad (39)$$

Hence, the following theorem has been proved.

**Theorem 3.1.** *If a smooth function  $U(x) = u(x)$  is an extremum of an action integral  $J[U] = \int_\Omega L[U]dx$  with  $L[U] = L(x, U, \partial U, \dots, \partial^k U)$ , then  $u(x)$  satisfies the Euler–Lagrange equations (39).*

### 3.2 Noether’s formulation of Noether’s theorem

We now present Noether’s formulation of her famous theorem. In this formulation, the action integral  $J[U]$  (35) is required to be invariant under the one-parameter Lie group of point transformations

$$\begin{aligned} (x^*)^i &= x^i + \varepsilon \xi^i(x, U) + O(\varepsilon^2), \quad i = 1, \dots, n, \\ (U^*)^\mu &= U^\mu + \varepsilon \eta^\mu(x, U) + O(\varepsilon^2), \quad \mu = 1, \dots, m, \end{aligned} \quad (40)$$

with corresponding infinitesimal generator given by

$$X = \xi^i(x, U) \frac{\partial}{\partial x^i} + \eta^\nu(x, U) \frac{\partial}{\partial U^\nu}, \quad (41)$$

Invariance holds if and only if  $\int_{\Omega^*} L[U^*]dx^* = \int_\Omega L[U]dx$  where  $\Omega^*$  is the image of  $\Omega$  under the point transformation (40). The Jacobian  $J$  of the transformation (40) is given by  $J = \det(D_i(x^*)^j) = 1 + \varepsilon D_i \xi^i(x, U) + O(\varepsilon^2)$ . Then  $dx^* = Jdx$ . Moreover, since (40) is a Lie group of transformations, it follows that  $L[U^*] = e^{\varepsilon X^{(k)}} L[U]$  in terms of the  $k$ th extension of the infinitesimal generator (41). Consequently, in Noether’s formulation, the one-parameter Lie group of point transformations (40) is a point symmetry of  $J[U]$  (35) if and only if

$$\int_\Omega (J e^{\varepsilon X^{(k)}} - 1) L[U] dx = \varepsilon \int_\Omega (L[U] D_i \xi^i(x, U) + X^{(k)} L[U]) dx + O(\varepsilon^2) \quad (42)$$

holds for arbitrary  $U(x)$  where  $X^{(k)}$  is the  $k$ th extended infinitesimal generator with  $U$  replacing  $u$ . Hence, if  $J[U]$  (35) has the point symmetry (40), then the  $O(\varepsilon)$  term in (42) vanishes, and thus one obtains the identity

$$L[U] D_i \xi^i(x, U) + X^{(k)} L[U] \equiv 0. \quad (43)$$

The one-parameter Lie group of point transformations (40) is equivalent to the one-parameter family of transformations

$$\begin{aligned} (x^*)^i &= x^i, \quad i = 1, \dots, n, \\ (U^*)^\mu &= U^\mu + \varepsilon [\eta^\mu(x, U) - U_i^\mu \xi^i(x, U)] + O(\varepsilon^2), \quad \mu = 1, \dots, m. \end{aligned} \quad (44)$$

Under the transformation (44), the corresponding infinitesimal change  $U(x) \rightarrow U(x) + \varepsilon v(x)$  has components  $v^\mu(x) = \hat{\eta}^\mu[U] = \eta^\mu(x, U) - U_i^\mu \xi^i(x, U)$  in terms of the transformations (44). Moreover, from the group property of (44), it follows that

$$\delta L = \varepsilon \hat{X}^{(k)} L[U] + O(\varepsilon^2) \quad (45)$$

where  $\hat{X}^{(k)}$  is the  $k$ th extension of the infinitesimal generator  $\hat{X} = \hat{\eta}^\mu[U] \frac{\partial}{\partial U^\mu}$  yielding the transformation (44). Thus

$$\int_{\Omega} \delta L dx = \varepsilon \int_{\Omega} \hat{X}^{(k)} L[U] dx + O(\varepsilon^2). \quad (46)$$

Consequently, after comparing expression (46) and expression (38) with  $v^\mu(x) = \hat{\eta}^\mu[U] = \eta^\mu(x, U) - U_i^\mu \xi^i(x, U)$ , it follows that

$$\hat{X}^{(k)} L[U] \equiv \hat{\eta}^\mu[U] E_{U^\mu}(L[U]) + D_l W^l[U, \hat{\eta}[U]] \quad (47)$$

where  $W^l[U, \hat{\eta}[U]]$  is given by expression (37) with the obvious substitutions.

The proof of the following theorem is obtained by direct calculation.

**Theorem 3.2.** *Let  $X^{(k)}$  be the  $k$ th extended infinitesimal generator of the one-parameter Lie group of point transformations (40) and let  $\hat{X}^{(k)}$  be the  $k$ th extended infinitesimal generator of the equivalent one-parameter family of transformations (44). Let  $F[U] = F(x, U, \partial U, \dots, \partial^k U)$  be an arbitrary function of its arguments. Then the following identity holds:*

$$X^{(k)} F[U] + F[U] D_i \xi^i(x, U) \equiv \hat{X}^{(k)} F[U] + D_i (F[U] \xi^i(x, U)). \quad (48)$$

Putting all of the above together, one obtains the following theorem.

**Theorem 3.3 (Noether's formulation of Noether's theorem).** *Suppose a given DE system  $\mathbf{R}\{x; u\}$  (1) is derivable from a variational principle, i.e., the given DE system is a set of Euler-Lagrange equations (39) whose solutions  $u(x)$  are extrema  $U(x) = u(x)$  of an action integral  $J[U]$  (35) with Lagrangian  $L[U]$ . Suppose the one-parameter Lie group of point transformations (40) is a point symmetry of  $J[U]$ . Let  $W^l[U, v]$  be defined by (37) for arbitrary functions  $U(x), v(x)$ . Then*

(1) *The identity*

$$\hat{\eta}^\mu[U] E_{U^\mu}(L[U]) \equiv -D_i (\xi^i(x, U) L[U] + W^i[U, \hat{\eta}[U]]) \quad (49)$$

*holds for arbitrary functions  $U(x)$ , i.e.,  $\{\hat{\eta}^\mu[U]\}_{\mu=1}^m$  is a set of local conservation law multipliers of the Euler-Lagrange system (39);*

(2) *The local conservation law*

$$D_i (\xi^i(x, u) L[u] + W^i[u, \hat{\eta}[u]]) = 0 \quad (50)$$

*holds for any solution  $u = \Theta(x)$  of the Euler-Lagrange system (39).*

**Proof.** Let  $F[U] = L[U]$  in the identity (48). Then from the identity (43), one obtains

$$\hat{X}^{(k)}L[U] + D_i(L[U]\xi^i(x, U)) \equiv 0 \quad (51)$$

holding for arbitrary functions  $U(x)$ . Substitution for  $\hat{X}^{(k)}L[U]$  in (51) through (47) yields (49). If  $U(x) = u(x)$  solves the Euler-Lagrange system (39), then the left-hand side of equation (49) vanishes. This yields the conservation law (50). ■

### 3.3 Boyer's formulation of Noether's theorem

Boyer [7] extended Noether's theorem to enable one to conveniently find conservation laws arising from invariance under higher-order transformations by generalizing Noether's definition of invariance of an action integral  $J[U]$  (35). In particular, under the following definition, an action integral  $J[U]$  (35) is invariant under a one-parameter higher-order local transformation if its integrand  $L[U]$  is invariant to within a divergence under such a transformation.

**Definition 3.1.** Let

$$\hat{X} = \hat{\eta}^\mu(x, U, \partial U, \dots, \partial^s U) \frac{\partial}{\partial U^\mu} \quad (52)$$

be the infinitesimal generator of a one-parameter higher-order local transformation

$$\begin{aligned} (x^*)^i &= x^i, \quad i = 1, \dots, n, \\ (U^*)^\mu &= U^\mu + \varepsilon \hat{\eta}^\mu(x, U, \partial U, \dots, \partial^s U) + O(\varepsilon^2), \quad \mu = 1, \dots, m, \end{aligned} \quad (53)$$

with its extension to all derivatives denoted by  $\hat{X}^\infty$ . Let

$$\hat{\eta}^\mu[U] = \hat{\eta}^\mu(x, U, \partial U, \dots, \partial^s U).$$

The transformation (53) is a *local symmetry* of  $J[U]$  (35) if and only if

$$\hat{X}^\infty L[U] \equiv D_i A^i[U] \quad (54)$$

holds for some set of functions  $A^i[U] = A^i(x, U, \partial U, \dots, \partial^r U)$ ,  $i = 1, \dots, n$ .

**Definition 3.2.** A local transformation with infinitesimal generator (52) that is a local symmetry of  $J[U]$  (35) is called a *variational symmetry* of  $J[U]$ .

The proof of the following theorem follows from the property of Euler operators annihilating divergences.

**Theorem 3.4.** A *variational symmetry with infinitesimal generator (52) of the action integral  $J[U]$  (35) yields a local symmetry with infinitesimal generator  $\hat{X} = \hat{\eta}^\mu(x, u, \partial u, \dots, \partial^s u) \frac{\partial}{\partial u^\mu}$  of the corresponding Euler-Lagrange system (39).*

The following theorem generalizes Noether's formulation of her theorem.

**Theorem 3.5 (Boyer's generalization of Noether's theorem).** *Suppose a given DE system  $\mathbf{R}\{x; u\}$  (1) is derivable from a variational principle, i.e., the given DE system is a set of Euler–Lagrange equations (39) whose solutions  $u(x)$  are extrema  $U(x) = u(x)$  of an action integral  $J[U]$  (35) with Lagrangian  $L[U]$ . Suppose a local transformation with infinitesimal generator (52) yields a variational symmetry of  $J[U]$ . Let  $W^i[U, v]$  be defined by (37) for arbitrary functions  $U(x), v(x)$ . Then*

(1) *The identity*

$$\hat{\eta}^\mu[U]E_{U^\mu}(L[U]) \equiv D_i(A^i[U] - W^i[U, \hat{\eta}[U]]) \quad (55)$$

*holds for arbitrary functions  $U(x)$ , i.e.,  $\{\hat{\eta}^\mu[U]\}_{\mu=1}^m$  is a set of local conservation law multipliers of the Euler-Lagrange system (39);*

(2) *The local conservation law*

$$D_i(W^i[u, \hat{\eta}[u]] - A^i[u]) = 0 \quad (56)$$

*holds for any solution  $u = \Theta(x)$  of the Euler-Lagrange system (39).*

**Proof.** For a local transformation with infinitesimal generator (52), it follows that the corresponding infinitesimal change  $U(x) \rightarrow U(x) + \varepsilon v(x)$  has components  $v^\mu(x) = \hat{\eta}^\mu[U]$ . Consequently, equation (45) becomes

$$\delta L = \varepsilon \hat{X}^\infty L[U] + O(\varepsilon^2).$$

But from (36) it follows that

$$\delta L = \varepsilon(\hat{\eta}^\mu[U]E_{U^\mu}(L[U]) + D_i(W^i[U, \hat{\eta}[U]])) + O(\varepsilon^2).$$

Hence it immediately follows that

$$\hat{X}^\infty L[U] = \hat{\eta}^\mu[U]E_{U^\mu}(L[U]) + D_i(W^i[U, \hat{\eta}[U]]) \quad (57)$$

holds for arbitrary functions  $U(x)$ . Since the local transformation with infinitesimal generator (52) is a variational symmetry of  $J[U]$  (35), it follows that equation (54) holds. Substitution for  $\hat{X}^\infty L[U]$  in (57) through (54) yields the identity (55). If  $U(x) = u(x)$  solves the Euler-Lagrange system (39), then the left-hand side of equation (55) vanishes. This yields the conservation law (56). ■

**Theorem 3.6.** *If a conservation law is obtained through Noether's formulation (Theorem 3.3), then the conservation law can be obtained through Boyer's formulation (Theorem 3.5).*

**Proof.** Suppose the one-parameter Lie group of point transformations (40) yields a conservation law. Then the identity (51) holds. Consequently,

$$\hat{X}^{(k)}L[U] = \hat{X}^\infty L[U] = D_i A^i[U] \quad (58)$$

where  $A^i[U] = -D_i(L[U]\xi^i(x, U))$ . But equation (58) is just the condition for the one-parameter Lie group of point transformations (40) to be a variational symmetry of  $J[U]$  (35). Consequently, one obtains the same conservation law from Boyer's formulation. ■

## 4 Limitations of Noether's theorem and consequent advantages of the direct method

There are several limitations inherent in using Noether's theorem to find local conservation laws for a given DE system  $\mathbf{R}\{x; u\}$ . First of all, it is restricted to variational systems. Consequently, the linearizing operator (Fréchet derivative) for  $\mathbf{R}\{x; u\}$ , *as written*, must be self-adjoint, which implies that  $\mathbf{R}\{x; u\}$  must be of even order (if it is a scalar PDE), and the number of PDEs must be the same as the number of dependent variables appearing in  $\mathbf{R}\{x; u\}$ . [In particular, this can be seen from comparing expressions (17) and (18).] In addition, one must find an explicit Lagrangian  $L[U]$  whose Euler–Lagrange equations yield  $\mathbf{R}\{x; u\}$ .

There is also the difficulty of finding the variational symmetries for a given variational DE system  $\mathbf{R}\{x; u\}$ . First, for the given DE system, one must determine local symmetries depending on derivatives of dependent variables up to some chosen order. Second, one must find an explicit Lagrangian  $L[U]$  and check if each symmetry of the given DE system leaves invariant the Lagrangian  $L[U]$  to within a divergence, i.e., if a symmetry is indeed a variational symmetry.

Finally, the use of Noether's theorem to find local conservation laws is coordinate dependent since the action of a point (contact) transformation can transform a DE having a variational principle to one that does not have one. On the other hand, it is known that conservation laws are *coordinate-independent* in the sense that a point (contact) transformation maps a conservation law into a conservation law [13], and therefore it follows that an ideal method for finding conservation laws should be coordinate-independent.

Artifices may make a given DE system variational. Such artifices include:

- *The use of multipliers.* As an example, the PDE  $u_{tt} + 2u_x u_{xx} + u_x^2 = 0$ , as written, does not admit a variational principle since its linearized equation  $v_{tt} + 2u_x v_{xx} + (2u_{xx} + 2u_x)v_x = 0$  is not self-adjoint. However, the equivalent PDE  $e^x[u_{tt} + 2u_x u_{xx} + u_x^2] = 0$ , as written, is self-adjoint!
- *The use of a contact transformation of the variables.* As an example, the PDE

$$e^x u_{tt} - e^{3x}(u + u_x)^2(u + 2u_x + u_{xx}) = 0, \quad (59)$$

as written, does not admit a variational principle, since its linearized PDE and the adjoint PDE are different. But the point transformation  $x^* = x$ ,  $t^* = t$ ,  $u^*(x^*, t^*) = y(x, t) = e^x u(x, t)$ , maps the PDE (59) into the self-adjoint PDE  $y_{tt} - (y_x)^2 y_{xx} = 0$ , which is the Euler-Lagrange equation for an extremum  $Y = y$  of the action integral with Lagrangian  $L[Y] = \frac{1}{2}Y_t^2 - \frac{1}{12}Y_x^4$ .

- *The use of a differential substitution.* As an example, the KdV equation (13) as written, obviously does not admit a variational principle since it is of odd order. But the well-known differential substitution  $u = v_x$  yields the related transformed KdV equation  $v_{xt} + v_x v_{xx} + v_{xxx} = 0$ , which arises from the Lagrangian  $L[V] = \frac{1}{2}V_{xx}^2 - \frac{1}{6}V_x^3 - \frac{1}{2}V_x V_t$ .

- *The use of an artificial additional equation.* For example, the linear heat equation  $u_t - u_{xx} = 0$  is not self-adjoint since its adjoint equation is given by  $w_t + w_{xx} = 0$ . However the decoupled PDE system  $u_t - u_{xx} = 0$ ,  $\tilde{u}_t + \tilde{u}_{xx} = 0$  is evidently self-adjoint! [In general, the formal system, obtained through appending any given DE system by the adjoint of its linearized system, is self-adjoint.]

The direct method for finding local conservation laws is free of all of the above problems. It is directly applicable to any DE system, whether or not it is variational. Moreover, it does not require the knowledge of a Lagrangian, whether or not one exists. Indeed, under the direct method, variational and non-variational DE systems are treated in the same manner.

The direct method is naturally coordinate-independent. This follows from the fact that a point (contact) transformation maps a conservation law into a conservation law, and hence either form of a conservation law (in original or transformed variables) will arise from corresponding sets of multipliers, which can be found by the direct method in either coordinate system.

Finding conservation laws through the direct method is computationally more straightforward than through Noether's theorem even when a given DE system is variational. One simply writes down the set of linear determining equations (5) holding for arbitrary functions  $U(x)$ , which in the case of a variational system, include the symmetry determining equations as a subset of the multiplier determining equations. Hence, the resulting linear determining equations for local multipliers are usually not as difficult to solve as those for local symmetries since this determining system is more over-determined in the variational case.

On the other hand, if a given DE system is variational and one has obtained the Lagrangian for the DE system, then it is worthwhile to combine the direct method with Noether's theorem as follows. First, use the direct method to find the local conservation law multipliers and hence the corresponding variational symmetries. Second, for each variational symmetry, find the corresponding divergence term  $D_i A^i[U]$  that arises from the use of Boyer's formulation of the extended Noether's theorem. Third, use expression (37) in conjunction with Boyer's formula (56) to find the resulting local conservation law.

Many examples illustrating the use of the direct method to find local conservation laws, including examples that compare the use of Noether's theorem and the direct method (for PDE systems that admit a variational formulation) appear in [11]. A comparison of the local symmetry and local conservation law structure for non-variational PDE systems appears in [11, 14].

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