On invariance properties of the wave equation

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A complete group classification is given of both the wave equation $c^2(x)u_{xx} - u_{tt} = 0$ (I) and its equivalent system $v_t = u_x$, $c^2(x)v_x = u_t$ (II) when the wave speed $c(x) \neq$const. Equations (I) and (II) admit either a two- or four-parameter group. For the exceptional case, $c(x) = (Ax + B)^2$, equation (I) admits an infinite group. Equations (I) and (II) do not always admit the same group for a given $c(x)$: The group for (I) can have more parameters or fewer parameters than that for (II); moreover, the groups can be different with the same number of parameters. Separately for (I) and (II), all possible $c(x)$ that admit a four-parameter group are found explicitly. The corresponding invariant (similarity) solutions are considered. Some of these wave speeds have realistic physical properties: $c(x)$ varies monotonically from one positive constant to another positive constant as x goes from $-\infty$ to $+\infty$.

I. INTRODUCTION

In this paper we consider invariance properties of second-order hyperbolic partial differential equations (PDE's) (wave equations)

$$c^2(x)u_{xx} - u_{tt} = 0 \tag{1.1}$$

and corresponding hyperbolic systems

$$v_t = u_x, \quad u_t = c^2(x)v_x. \tag{1.2}$$

Their invariance properties are used to construct solutions of these PDE's for various classes of wave speeds $c(x)$.

An important related equation is

$$(c^2(x)v_x)_x - v_{tt} = 0. \tag{1.3}$$

Many physical problems lead to (1.1)–(1.3). Equation (1.1) arises in the study of small transverse vibrations of a string with variable density, system (1.2) in the study of transmission lines with variable capacitance or variable resistance, and Eq. (1.3) in the study of small longitudinal vibrations of a rod with variable Young's modulus.

Equations (1.1)–(1.3) are equivalent in the following senses [(1.4)–(1.7)]:

if $\{u(x,t), v(x,t)\}$ satisfy (1.2),
then $u(x,t)$ solves (1.1)
and $v(x,t)$ solves (1.3); (1.4)

if $u = F(x,t)$ satisfies (1.1),
then $(u,v) = (F_t,F_x)$ solves (1.2)
and $v = F_x$ solves (1.3); (1.5)

if $v = G(x,t)$ satisfies (1.3),
then $(u,v) = (c^2(x)G_x,G_t)$ solves (1.2)
and $u = c^2(x)G_x$ solves (1.1). (1.6)

Under the transformation

$$y = \int c^2(x)dx,$$

Eq. (1.3) can be rewritten as an equation of the form (1.1), namely,

$$c^{-1}(x(y))v_{yy} - v_{tt} = 0. \tag{1.7}$$

In spite of the apparent equivalence of a single PDE and a corresponding system of PDE's it does not necessarily follow that their respective invariance groups of point transformation are the same. It could happen that the group of point transformations leaving invariant the system is larger than that leaving invariant the single equation; also the converse could be true. We will show that this is indeed the case for the single equation (1.1) and the corresponding system (1.2). For example we show that if $c(x) = (Ax + B)^2$, then (1.1) is invariant under an infinite Lie group of point transformations, whereas the Lie group of point transformations leaving invariant (1.2) has only four parameters; if $c(x) = \sqrt{A + Be^{kx}}$, then the Lie group of (1.1) has two parameters and that of (1.2) has four parameters.

Consequently it follows that invariant (similarity) solutions of a system of PDE's lead to noninvariant solutions of a corresponding equivalent single PDE and vice versa. In Sec. IV of this paper we construct such noninvariant solutions for (1.1).

It is important to note that under the hodograph transformation (the interchange of dependent and independent variables), system (1.2) is equivalent to the nonlinear system

$$v_t = u_x, \; u_t = c^2(v)v_x. \tag{1.8}$$

Consequently if $\{u(x,t),v(x,t)\}$ solve (1.8) then $v(x,t)$ solves

$$(c^2(v)v_x)_x - v_{tt} = 0, \tag{1.9}$$

and introducing the potential $\phi(x,t)$, where $(u,v) = (\phi_t,\phi_x)$, the system (1.8) reduces to

$$c^2(\phi_x)\phi_{xx} - \phi_{tt} = 0. \tag{1.10}$$

The rest of this paper is organized as follows.

In Sec. II the Lie group of point transformations admitted by (1.1) is derived for all possible wave speeds $c(x)$. The corresponding invariant solutions are constructed.

In Sec. III the Lie group of point transformations ad-

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mitted by (1.2) is derived for all possible $c(x)$. If $c(x)$ satisfies the ordinary differential equation

$$cc'(c/c')'' = \pm\lambda^2, \quad \lambda \neq 0, \tag{1.11}$$

then (1.2) admits a larger group than (1.1). (Throughout this paper a prime denotes differentiation of a function of a single variable.) Invariant solutions of (1.2) and hence solutions of (1.1) are constructed for $c(x)$ satisfying (1.11).

In Sec. IV we discuss the differences between the invariance properties of the single equation (1.1) and the system (1.2). We show that in general the Lie group of point transformation leaving invariant (1.2) [(1.1)] does not necessarily correspond to a Lie group of point transformations or Lie–Bäcklund transformations leaving invariant (1.1) [(1.2)].

In Sec. V we find the equivalence classes of wave speeds $c(x)$ for the wave equation (1.1).

II. THE INVARIANCE PROPERTIES OF THE WAVE EQUATION AS A SINGLE EQUATION

Lie[1,2] proved that a second-order linear hyperbolic PDE with two independent variables admits a group of point transformations containing at most four parameters if it does not admit an infinite group. Lie did not study specifically the wave equation (1.1).

A. Infinitesimal transformations

By using Lie's algorithm,[2,3] one can find the generators of the invariance group of point transformations of (1.1). If the point transformation

$$X = x + \epsilon\xi(x,t) + O(\epsilon^2),$$
$$T = t + \epsilon\tau(x,t) + O(\epsilon^2), \tag{2.1}$$
$$U = u + \epsilon f(x,t)u + O(\epsilon^2),$$

leaves (1.1) invariant, then its infinitesimals $\{\xi,\tau,f\}$ satisfy the *determining equations*

$$\xi_t - c^2(x)\tau_x = 0; \tag{2.2a}$$

$$c(x)[\tau_t - \xi_x] + c'(x)\xi = 0; \tag{2.2b}$$

$$\tau_{tt} - 2f_t - c^2(x)\tau_{xx} = 0; \tag{2.2c}$$

$$\xi_{tt} + c^2(x)[2f_x - \xi_{xx}] = 0; \tag{2.2d}$$

$$f_{tt} - c^2(x)f_{xx} = 0. \tag{2.2e}$$

Solving (2.2a) for τ_x and (2.2b) for τ_t and setting $\tau_{tx} = \tau_{xt}$, one finds that

$$\xi_{xx} - (1/c^2)\xi_{tt} - [H(x)\xi]_x = 0, \tag{2.3}$$

where $H(x) = c'/c$.

The solution of (2.3), (2.2c), and (2.2d) for f leads to

$$f = \tfrac{1}{2}H\xi + s, \quad s = \text{const}. \tag{2.4}$$

Substituting (2.4) into (2.2e), one obtains

$$[(2H' + H^2)\xi^2]_x = 0. \tag{2.5}$$

From Eq. (2.5) there follow three cases.

Case I: $2H' + H^2 = 0$

In this case

$$c(x) = (Ax + B)^2, \tag{2.6}$$

where A and B are arbitrary constants. It is easy to show that here an infinite group leaves invariant (1.1). In particular for *any* solution $\xi(x,t)$ of the corresponding equation (2.3), one can find $\{\tau(x,t), f(x,t)\}$ solving (2.2a)–(2.2e),

$$\tau = \int(\xi_x - H\xi)dt, \quad f = \frac{A\xi(x,t)}{Ax + B}. \tag{2.7}$$

Case II: $2H' + H^2 \neq 0, \xi \neq 0$

From (2.5) it follows that ξ can be expressed in the separable form

$$\xi(x,t) = \alpha(x)\beta(t), \tag{2.8}$$

where

$$\alpha^2(x) = [2H' + H^2]^{-1} \tag{2.9}$$

and $\beta(t)$ is to be determined.

Substituting (2.4) and (2.8) into (2.2d), one finds that

$$\frac{\beta''(t)}{\beta(t)} = \frac{c^2(\alpha' - H\alpha)'}{\alpha} = \text{const} = \sigma^2. \tag{2.10}$$

Note that α, σ could be real or imaginary.

Case II(a): The subcase $\sigma = 0$

Here $c(x)$ must satisfy the differential equation

$$(\alpha' - H\alpha)' = 0 \tag{2.11}$$

and correspondingly

$$\beta(t) = p + qt, \tag{2.12}$$

where p and q are arbitrary constants.

The substitution of (2.4) and (2.8) into (2.2e) leads to

$$(\alpha H)'' = 0. \tag{2.13}$$

Thus it is necessary and sufficient that the wave speed $c(x)$ satisfy Eqs. (2.11) and (2.13). The general solution of these equations is

$$\alpha = Bx^2 + Cx + D, \tag{2.14}$$
$$\alpha H = A + 2Bx, \tag{2.15}$$

where $\{A, B, C, D\}$ are arbitrary constants. Consequently

$$c(x) = (Bx^2 + Cx + D) \times \exp\left((A - C)\int(Bx^2 + Cx + D)^{-1}dx\right). \tag{2.16}$$

It is easy to show that

$$\tau = (C - A)\left(pt + \frac{1}{2}qt^2\right) + q\int\frac{\alpha}{c^2}dx + r, \tag{2.17}$$

where r is another arbitrary constant.

If $B = 0$ in Eq. (2.16), then this expression reduces to the general form

$$c(x) = (Ax + B)^C, \tag{2.18}$$

where $\{A, B, C\}$ are arbitrary constants, $C \neq 0, 2$.

If $B = C = 0$ in (2.16), then the corresponding wave speeds are of the general form

$$c(x) = Ae^{Bx}, \tag{2.19}$$

where A and B are arbitrary constants.

Case II(b): The subcase $\sigma \neq 0$

Here Eq. (2.10) leads to $c(x)$ solving

$$c^2(\alpha' - H\alpha)' = \sigma^2\alpha, \tag{2.20}$$

where $H = c'/c$ and α is given by (2.9). Equation (2.20) can be integrated to give

$$(\alpha' - H\alpha)^2 - (\sigma\alpha/c)^2 = \text{const} = K. \tag{2.21}$$

$\beta(t)$ solves $\beta'' = \sigma^2\beta$, i.e., $\beta = pe^{\sigma t} + qe^{-\sigma t}$.

Thus in this subcase the infinitesimals of (2.1) become

$$\xi = \alpha(x)[pe^{\sigma t} + qe^{-\sigma t}],$$
$$\tau = \sigma^{-1}[\alpha' - H\alpha][pe^{\sigma t} - qe^{-\sigma t}] + r, \qquad (2.22)$$
$$f = \tfrac{1}{2}\alpha H[pe^{\sigma t} + qe^{-\sigma t}] + s,$$

where the group parameters $\{p, q, r, s\}$ are arbitrary constants. The solution of Eq. (2.9), (2.21) for the wave speed $c(x)$ is given in Appendix A. In Case II, if $\xi \neq 0$, the wave equation (1.1) is invariant under a four-parameter Lie group of point transformations.

Case III: $\xi = 0$

From the determining equations (2.2a)–(2.2e) it follows immediately that

$$\tau = \text{const} = r, \quad f = \text{const} = s,$$

and hence (1.1) is invariant only under translations in t and scalings of u. In particular for any wave speed $c(x)$ that does not solve the system (2.9), (2.20) for any σ (zero or nonzero), the wave equation (1.1) is invariant only under this trivial two-parameter Lie group of point transformations.

Hence the following theorem has been proved.

Theorem: The wave equation (1.1), whose wave speed $c(x)$ is a solution of system (2.9), (2.21) for any σ (zero or nonzero), is invariant under a four-parameter Lie group of point transformations. The group becomes infinite if and only if $c(x) = (Ax+B)^2$. All other wave speeds $c(x)$ admit the two-parameter group of translations in t and scalings of u.

B. Group generators and their Lie algebras in the finite parameter cases

If (2.1) leaves invariant (1.1), the corresponding group generator is

$$L = \xi(x,t)\frac{\partial}{\partial x} + \tau(x,t)\frac{\partial}{\partial t} + f(x,t)u\frac{\partial}{\partial u}. \qquad (2.23)$$

To the parameters $\{p, q, r, s\}$ of the group there correspond generators $\{L_p, L_q, L_r, L_s\}$. The generators form a Lie algebra. The generators for all possible wave speeds $c(x)$ follow. Cases (i)–(iv) relate to $\sigma = 0$.

Case (i): $c(x) = (Bx^2 + Cx + D)\exp((A-C)\int(Bx^2 + Cx + D)^{-1}dx)$

Here

$$L_p = [Bx^2 + Cx + D]\frac{\partial}{\partial x} + [C - A]t\frac{\partial}{\partial t} + \frac{1}{2}[A + 2Bx]u\frac{\partial}{\partial u},$$
$$L_q = t[Bx^2 + Cx + D]\frac{\partial}{\partial x} + \left[\frac{1}{2}(C-A)t^2 + \int\frac{Bx^2 + Cx + D}{c^2(x)}dx\right]\frac{\partial}{\partial t} + \frac{1}{2}t[A + 2Bx]u\frac{\partial}{\partial u}, \qquad (2.24)$$
$$L_r = \frac{\partial}{\partial t}, \quad L_s = u\frac{\partial}{\partial u}.$$

The commutator table for the Lie algebra is

$$[L_p, L_q] = (C-A)L_q; \quad [L_p, L_r] = (A-C)L_r;$$
$$[L_q, L_r] = -L_p; \quad \begin{bmatrix} L_p, L_s \\ q \\ r \end{bmatrix} = 0. \qquad (2.25)$$

It is easy to show that this group is isomorphic to SO(2,1) when $A - C \neq 0$. An interesting special case is $A = C$ where $c(x) = Bx^2 + Cx + D$.

Case (ii): $c(x) = (Ax+B)^C$, $C \neq 0,1,2$

Here

$$L_p = (Ax+B)\frac{\partial}{\partial x} + A(1-C)t\frac{\partial}{\partial t} + \frac{1}{2}ACu\frac{\partial}{\partial u},$$
$$L_q = (Ax+B)t\frac{\partial}{\partial x} + \frac{1}{2}\left[A(1-C)t^2 + \frac{(Ax+B)^{2-2C}}{A(1-C)}\right]\frac{\partial}{\partial t} + \frac{1}{2}ACtu\frac{\partial}{\partial u}, \qquad (2.26)$$
$$L_r = \frac{\partial}{\partial t}, \quad L_s = u\frac{\partial}{\partial u}.$$

The commutator table for the Lie algebra is the same as (2.25) with $(C-A)$ replaced by $A(1-C)$.

Case (iii): $c(x) = Ax + B$

Here

$$L_p = (Ax+B)\frac{\partial}{\partial x} + \frac{1}{2}Au\frac{\partial}{\partial u},$$
$$L_q = (Ax+B)t\frac{\partial}{\partial x} + \left[\frac{1}{A}\log(Ax+B)\right]\frac{\partial}{\partial t} + \frac{1}{2}Atu\frac{\partial}{\partial u}, \qquad (2.27)$$
$$L_r = \frac{\partial}{\partial t}, \quad L_s = u\frac{\partial}{\partial u}.$$

The corresponding commutator table is

$$[L_p, L_q] = L_r; \quad [L_p, L_r] = 0;$$
$$[L_q, L_r] = -L_p; \quad \begin{bmatrix} L_p, L_s \\ q \\ r \end{bmatrix} = 0.$$

Case (iv): $c(x) = Ae^{Bx}$

Here

$$L_p = A\frac{\partial}{\partial x} - ABt\frac{\partial}{\partial t} + \frac{1}{2}ABu\frac{\partial}{\partial u},$$
$$L_q = At\frac{\partial}{\partial x} - \frac{1}{2}\left[ABt^2 + \frac{1}{AB}e^{-2Bx}\right]\frac{\partial}{\partial t} + \frac{1}{2}ABtu\frac{\partial}{\partial u}, \qquad (2.28)$$
$$L_r = \frac{\partial}{\partial t}, \quad L_s = u\frac{\partial}{\partial u}.$$

The commutator table is the same as (2.25) with $A - C$ replaced by AB.

Cases (ii)–(iv) can result as limiting cases for the constants $\{A, B, C, D\}$ of case (i).

Case (v): $c(x)$ for $\sigma \neq 0$

From (2.22)

$$L_p = e^{\sigma t}\left[\alpha \frac{\partial}{\partial x} + \sigma^{-1}(\alpha' - H\alpha)\frac{\partial}{\partial t} + \frac{1}{2}\alpha H u \frac{\partial}{\partial u}\right],$$

$$L_q = e^{-\sigma t}\left[\alpha \frac{\partial}{\partial x} - \sigma^{-1}(\alpha' - H\alpha)\frac{\partial}{\partial t} + \frac{1}{2}\alpha H u \frac{\partial}{\partial u}\right], \qquad (2.29)$$

$$L_r = \frac{\partial}{\partial t}, \quad L_s = u\frac{\partial}{\partial u}.$$

The corresponding commutator table is

$$[L_p, L_q] = 2\sigma^{-1}KL_r; \quad [L_p, L_r] = -\sigma L_p;$$
$$[L_q, L_r] = \sigma L_q; \quad \begin{bmatrix} L_p, L_s \\ q \\ r \end{bmatrix} = 0. \qquad (2.30)$$

Recall that K is given by (2.21).

Clearly this group is isomorphic to SO(2,1) when $K \neq 0$. When σ is imaginary, appropriate linear combinations of L_p and L_q will yield the corresponding real Lie algebra.

Case (iv): All other $c(x)$

Here the generators are only

$$L_r = \frac{\partial}{\partial t}; \quad L_s = u\frac{\partial}{\partial u}. \qquad (2.31)$$

C. The infinite group case: $c(x) = (Ax+B)^2$

In this case the wave equation (1.1) becomes

$$(Ax+B)^2 u_{xx} - u_{tt} = 0. \qquad (2.32)$$

Equation (2.32) can be mapped into the wave equation $(A \neq 0)$

$$U_{XT} = 0 \qquad (2.33)$$

by the transformation[4]

$$X = [1/(Ax+B)] + At,$$
$$T = [1/(Ax+B)] - At, \qquad (2.34)$$
$$U = (Ax+B)^{-1}u.$$

Hence the general solution of (2.32) is

$$u = (Ax+B)[F(X) + G(T)], \qquad (2.35)$$

where F and G are arbitrary twice differentiable functions of their respective arguments.

D. Similarity solutions of the wave equation (1.1)

A *similarity solution* (invariant solution)[2,3] of (1.1) is a solution $u = \theta(x,t)$ of (1.1) satisfying the characteristic equations

$$\frac{dx}{\xi(x,t)} = \frac{dt}{\tau(x,t)} = \frac{du}{f(x,t)u}, \qquad (2.36)$$

corresponding to an admitted group (2.1). The *similarity variable* $z(x,t)$ is the constant of integration of the first equality of (2.36).

For all of our cases, similarity solutions for $r \neq 0$ can always be obtained from similarity solutions for $r = 0$ by replacing t by $t + r$. For the cases where $\sigma = 0$, the class of similarity solutions for $\{q = 1, r \text{ arbitrary}, s \text{ arbitrary}, p = 0\}$ is identical to the class of similarity solutions for $\{q = 1, p, r, s, \text{ arbitrary}\}$ since the commutator of L_q with L_r generates L_p. Next we discuss similarity solutions of (1.1) keeping in mind the above remarks.

Case (i): Similarity solutions of (1.1) for $p = q = 0$, $r = 1$, s arbitrary

Here (2.36) becomes

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{su}. \qquad (2.37)$$

The similarity variable $z = x$, and the similarity form for the similarity solutions is

$$u = e^{st}F(x;s), \qquad (2.38)$$

where $F(x;s)$ is a function of x and the parameter s. Substituting (2.38) into (1.1), one find that $F(x;s)$ satisfies the ordinary differential equation (ODE)

$$c^2(x)F_{xx}(x;s) - s^2F(x;s) = 0, \qquad (2.39)$$

If $\{F_1(x;s), F_2(x;s)\}$ are linearly independent solutions of (2.39) for any s, then any linear superposition

$$u = \sum_s e^{st}[A_1(s)F_1(x;s) + A_2(s)F_2(x;s)] \qquad (2.40)$$

solves (1.1) for arbitrary $\{A_1(s), A_2(s)\}$. Note that the sum in (2.40) can be replaced by an integral with respect to s.

Now we consider all cases for invariance of (1.1) under a four-parameter group. The following cases (ii)–(v) correspond to $\sigma = 0$ in Eq. (2.10).

Case (ii): $c(x) = x^C$, $C \neq 0,1,2$

The substitutions $Ax + B \to x$, $t \to A^{-1}t$, make the PDE

$$(Ax+B)^{2C}u_{xx} - u_{tt} = 0 \qquad (2.41)$$

equivalent to the PDE

$$x^{2C}u_{xx} - u_{tt} = 0. \qquad (2.42)$$

1. Similarity solutions of (2.42) for $q = r = 0$, $p = 1$, s arbitrary

Here (2.36) becomes equivalently

$$\frac{dx}{x} = \frac{dt}{(1-C)t} = \frac{du}{su}. \qquad (2.43)$$

The similarity variable is

$$z = x^{C-1}t. \qquad (2.44)$$

The similarity form for the solutions is

$$u = x^sF(z;s). \qquad (2.45)$$

$F(z;s)$ satisfies the ODE

$$[1 - (C-1)^2z^2]F_{zz}(z;s) + (1-C)(s+C-1)zF_z(z;s) + s(1-s)F(z;s) = 0. \qquad (2.46)$$

Linearly independent solutions of (2.46) are

$$F_1(z;s) = F(\alpha,\beta;\gamma;\zeta)$$

and (2.47)

$$F_2(z;s) = \zeta^{1-C}F(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma;\zeta),$$

where $F(\alpha,\beta;\gamma;\zeta)$ is the hypergeometric function,

$$\alpha=\frac{s}{1-C},\quad \beta=\frac{s-1}{1-C},\quad \gamma=\frac{1}{2}\frac{(2s+C-2)}{C-1},$$
$$\zeta=\frac{1}{2}+\frac{1}{2}(C-1)z. \tag{2.48}$$

2. Similarity solutions of (2.42) for $p=r=0$, $q=1$, s arbitrary

In this case (2.36) is equivalent to

$$\frac{dx}{2tx}=\frac{dt}{[x^{2-2C}/(1-C)]+(1-C)t^2}=\frac{du}{(Ct+[s/(C-1)])u}. \tag{2.49}$$

The similarity variable is

$$z=(C-1)^2t^2x^{C-1}-x^{1-C}. \tag{2.50}$$

The similarity solutions are of the form

$$u=x^{C/2}e^{sx^{C-1}t/z}F(z;s). \tag{2.51}$$

$F(z;s)$ satisfies the ODE

$$4(C-1)^2[z^2F_{zz}(z;s)+2zF_z(z;s)]+[C(C-2)-4sz^{-2}]F(z;s)=0. \tag{2.52}$$

If $[1/(C-1)]\neq$integer, then linearly independent solutions of (2.52) are

$$F(z;s)=z^{-1/2}I_{\pm\nu}(\zeta), \tag{2.53}$$

where $I_\nu(\zeta)$ is a modified Bessel function of order ν,

$$\nu=\frac{1}{2(C-1)},\quad \zeta=\frac{sz^{-1}}{C-1}. \tag{2.54}$$

Case (iii): $c(x)=x$

Here we consider the PDE

$$(Ax+B)^2u_{xx}-u_{tt}=0 \tag{2.55}$$

equivalent to the PDE

$$x^2u_{xx}-u_{tt}=0. \tag{2.56}$$

3. Similarity solutions of (2.56) for $q=r=0$, $p=1$, s arbitrary

The characteristic equations (2.36) are equivalently

$$\frac{dx}{x}=\frac{dt}{0}=\frac{du}{su}.$$

The similarity variable is

$$z=t \tag{2.57}$$

with corresponding similarity form

$$u=x^sF(t;s). \tag{2.58}$$

$F(t;s)$ satisfies the ODE

$$F_{tt}(t;s)+s(1-s)F(t;s)=0. \tag{2.59}$$

The resulting superposition of similarity solutions is

$$u(x,t)=\sum_s x^s[A_1(s)e^{\sqrt{s(s-1)}t}+A_2(s)e^{-\sqrt{s(s-1)}t}]. \tag{2.60}$$

These solutions are of the form (2.40).

4. Similarity solutions of (2.56) for $p=r=0$, $q=1$, s arbitrary

Now (2.36) is equivalently

$$\frac{dx}{2tx}=\frac{dt}{2\log x}=\frac{du}{(t+2s)u}. \tag{2.61}$$

The similarity variable is

$$z=t^2-(\log x)^2. \tag{2.62}$$

The corresponding form of the similarity solutions is

$$u=x^{1/2}|\log x+t|^sF(z;s). \tag{2.63}$$

$F(z;s)$ satisfies the ODE

$$16z^2F_{zz}(z;s)+16(1+s)zF_z(z;s)-zF(z;s)=0. \tag{2.64}$$

If $2s\neq$integer, linearly indepedent solutions of (2.64) are

$$F(z;s)=z^{s/2}I_{\pm\nu}(\zeta), \tag{2.65}$$

where

$$\nu=s,\quad \zeta=\tfrac{1}{4}z^{1/2}. \tag{2.66}$$

Case (iv): $c(x)=e^{-x/2}$

The substitutions $x\to-x/2B$, $t\to t/2AB$, make the PDE

$$A^2e^{2Bx}u_{xx}-u_{tt}=0 \tag{2.67}$$

equivalent to the PDE

$$e^{-x}u_{xx}-u_{tt}=0. \tag{2.68}$$

5. Similarity solutions of (2.68) for $q=r=0$, $p=1$, s arbitrary

The characteristic equations (2.36) are equivalent to

$$\frac{dx}{2}=\frac{dt}{t}=\frac{du}{2su}. \tag{2.69}$$

The similarity variable is

$$z=te^{-x/2}. \tag{2.70}$$

The similarity solutions are of the form

$$u=e^{sx}F(z;s). \tag{2.71}$$

$F(z;s)$ satisfies the ODE

$$(4-z^2)F_{zz}(z;s)+(4s-1)zF_z(z;s)-4s^2F(z;s)=0. \tag{2.72}$$

Linearly independent solutions of (2.72) are of the hypergeometric form (2.47), where

$$\alpha=\beta=-2s,\quad \gamma=\tfrac{1}{2}(1-4s), \tag{2.73}$$

and

$$\zeta=\tfrac{1}{2}+\tfrac{1}{4}z. \tag{2.74}$$

6. Similarity solutions of (2.68) for $p=r=0$, $q=1$, s arbitrary

Now (2.36) is equivalent to

$$\frac{dx}{4t}=\frac{dt}{t^2+4e^x}=\frac{du}{(s-t)u}. \tag{2.75}$$

The similarity variable is

$$z=t^2e^{-x/2}-4e^{x/2}. \tag{2.76}$$

The corresponding similarity solutions are of the form

$$u=\exp(-[\tfrac{1}{4}x+stz^{-1}e^{-x/2}])F(z;s). \qquad (2.77)$$

$F(z;s)$ satisfies the ODE

$$4z^2F_{zz}(z;s)+8zF_z(z;s)+(1-16s^2z^{-2})F(z;s)=0. \qquad (2.78)$$

This equation has linearly independent solutions

$$F_1(z;s)=z^{-1/2}I_0(\zeta),\quad F_2(z;s)=z^{-1/2}K_0(\zeta), \qquad (2.79)$$

where $\{I(\zeta),K(\zeta)\}$ are modified Bessel functions of order 0, and

$$\zeta=2sz^{-1}. \qquad (2.80)$$

Case (v): $c(x)=(Bx^2+Cx+D)\exp((A-C)\int(Bx^2+Cx+D)^{-1}dx)$

By appropriate scalings and translations in x and scalings in t, the corresponding wave equation (1.1) is equivalent to one of the five canonical forms (2.42), (2.68), or

$$[(x^2+1)^2e^{4A\arctan x}]u_{xx}-u_{tt}=0, \qquad (2.81)$$

$$[(1-x)^{2+2A}(1+x)^{2-2A}]u_{xx}-u_{tt}=0, \qquad (2.82)$$

$$[x^4e^{2/x}]u_{xx}-u_{tt}=0. \qquad (2.83)$$

In Eqs. (2.81), (2.82), A is an arbitrary constant.

Case (va): $c(x)=(x^2+1)e^{2A\arctan x}$

7. Similarity solutions of (2.81) for $q=r=0$, $p=1$, s arbitrary

The characteristic equations (2.36) are

$$\frac{dx}{1+x^2}=\frac{dt}{-2At}=\frac{du}{(x+s)u}. \qquad (2.84)$$

The similarity variable is

$$z=te^{2Ay}, \qquad (2.85)$$

where

$$y=\arctan x. \qquad (2.86)$$

The corresponding similarity form is

$$u=\sqrt{1+x^2}e^{sy}F(z;s). \qquad (2.87)$$

$F(z;s)$ solves the ODE

$$(4A^2z^2-1)F_{zz}(z;s)+4A(A+s)zF_z(z;s)+(1+s^2)F(z;s)=0 \qquad (2.88)$$

whose general solution can be expressed in terms of hypergeometric functions.

In the special case $A=0$, the resulting superposition of similarity solutions is

$$u(x,t)=\sqrt{x^2+1}\sum_s e^{sy}[A_1(s)e^{\sqrt{s^2+1}t}+A_2(s)e^{-\sqrt{s^2+1}t}]. \qquad (2.89)$$

These solutions are of the form (2.40).

8. Similarity solutions of (2.81) for $p=0$, $r=1/4A$, $q=1$, s arbitrary

The characteristic equations are

$$\frac{dx}{t(1+x^2)}=\frac{4A\,dt}{-4A^2t^2-e^{-4Ay}+1}=\frac{du}{[t(A+x)+s]u}. \qquad (2.90)$$

The similarity variable is

$$z=2A^2t^2e^{2Ay}-\cosh 2Ay, \qquad (2.91)$$

where

$$y=\arctan x. \qquad (2.92)$$

The resulting similarity form is

$$u=\sqrt{1+x^2}e^{Ay}|z+e^{2Ay}(1+2At)|^sF(z;s). \qquad (2.93)$$

$F(z;s)$ satisfies the ODE

$$4A^2(z^2-1)F_{zz}(z;s)+8A^2(1+s)zF_z(z;s)+\{1+[A(1+2s)]^2\}F(z;s)=0. \qquad (2.94)$$

Linearly independent solutions of (2.94) are of the hypergeometric form (2.47), where

$$\alpha=\frac{1}{2}+s+\frac{i}{2A},\quad \beta=\frac{1}{2}+s-\frac{i}{2A},\quad \gamma=1+s,\quad \zeta=\tfrac{1}{2}(1+z). \qquad (2.95)$$

In the special case $A=0$, the similarity variable becomes

$$z=-t^2+y^2. \qquad (2.96)$$

Here the similarity form reduces to

$$u=\sqrt{x^2+1}(t+\arctan x)^sF(z;s). \qquad (2.97)$$

$F(z;s)$ satisfies the ODE

$$4zF_{zz}(z;s)+4(s+1)F_z(z;s)+F(z;s)=0. \qquad (2.98)$$

Solutions of (2.98) can be expressed in terms of Bessel functions:

$$F(z;s)=z^{-s/2}J_{\pm\nu}(\zeta), \qquad (2.99)$$

where

$$\nu=s,\quad \zeta=z^{-1/2}. \qquad (2.100)$$

Case (vb): $c(x)=(1-x)^{1+A}(1+x)^{1-A}$

9. Similarity solutions of (2.82) for $q=r=0$, $p=1$, s arbitrary

The characteristic equations are equivalent to

$$\frac{dx}{x^2-1}=\frac{dt}{-2At}=\frac{du}{(x+2s)u}. \qquad (2.101)$$

The similarity variable is

$$z=ty^A, \qquad (2.102)$$

where

$$y=(1-x)/(1+x). \qquad (2.103)$$

The similarity form is

$$u=\sqrt{1-x^2}y^sF(z;s). \qquad (2.104)$$

$F(z;s)$ satisfies the ODE

$$(4A^2z^2-1)F_{zz}(z;s)+4A(A+2s)zF_z(z;s)$$
$$+(4s^2-1)F(z;s)=0. \quad (2.105)$$

Linearly independent solutions of (2.105) are of the hypergeometric form (2.47), where

$$\alpha=\frac{1}{A}\left[2s-\frac{1}{2}\right],\quad \beta=\frac{1}{2}A,$$
$$\gamma=\frac{1}{2}+\frac{s}{A},\quad \zeta=\frac{1}{2}+Az. \quad (2.106)$$

In the special case $A=0$, the resulting superposition of similarity solutions, which is of the form (2.40), is

$$u=\sqrt{1-x^2}\sum_s y^s[A_1(s)e^{\sqrt{4s^2-1}t}+A_2(s)e^{-\sqrt{4s^2-1}t}]. \quad (2.107)$$

10. Similarity solutions of (2.82) for p=0, r=1/A, q=1, s arbitrary

Here the characteristic equations are

$$\frac{dx}{(x^2-1)t}=\frac{4A\,dt}{1-4A^2t^2-y^{-2A}}=\frac{du}{[(A+x)t+s]u}. \quad (2.108)$$

The similarity variable is

$$z=2A^2t^2y^A-\tfrac{1}{2}(y^A+y^{-A}), \quad (2.109)$$

where

$$y=(1-x)/(1+x). \quad (2.110)$$

The resulting similarity solutions are of the form

$$u=\sqrt{1-x^2}y^{A/2}|(1+2At)y^A+z|^sF(z;s). \quad (2.111)$$

$F(z;s)$ satisfies the ODE

$$4A^2(z^2-1)F_{zz}(z;s)+8A^2(s+1)zF_z(z;s)$$
$$+[A^2(2s+1)^2-1]F(z;s)=0. \quad (2.112)$$

Linearly independent solutions of (2.112) are of the hypergeometric form (2.47), where

$$\alpha=s+\frac{1}{2}+\frac{1}{2A},\quad \beta=s+\frac{1}{2}-\frac{1}{2A},$$
$$\gamma=s+1,\quad \zeta=\tfrac{1}{2}(z+1). \quad (2.113)$$

Case (vc): $c(x)=x^2e^{1/x}$

11. Similarity solutions of (2.83) for q=r=0, p=1, s arbitrary

The characteristic equations are equivalent to

$$\frac{dx}{x^2}=\frac{dt}{t}=\frac{du}{(x-s)u}. \quad (2.114)$$

The similarity variable is

$$z=te^{1/x} \quad (2.115)$$

and the corresponding similarity form is

$$u=xe^{s/x}F(z;s). \quad (2.116)$$

$F(z;s)$ solves the ODE

$$(z^2-1)F_{zz}(z;s)+(2s+1)zF_z(z;s)+s^2F(z;s)=0. \quad (2.117)$$

Linearly independent solutions of (2.117) are of the hypergeometric form (2.47) where

$$\alpha=\beta=s,\quad \gamma=\tfrac{1}{2}+s,\quad \zeta=\tfrac{1}{2}(1+z). \quad (2.118)$$

12. Similarity solutions of (2.83) for p=r=0, q=1, s arbitrary

Here the characteristic equations are

$$\frac{dx}{2tx^2}=\frac{dt}{t^2+e^{-2/x}}=\frac{du}{[(2x-1)t+2s]u}. \quad (2.119)$$

The similarity variable is

$$z=t^2e^{1/x}-e^{-1/x}, \quad (2.120)$$

and the resulting similarity form is

$$u=xe^{1/2x}e^{-2ste^{1/x}z^{-1}}F(z;s). \quad (2.121)$$

$F(z;s)$ satisfies the ODE

$$4z^2F_{zz}(z;s)+8zF_z(z;s)+(1-16s^2z^{-2})F(z;s)=0. \quad (2.122)$$

Linearly independent solutions of (2.122) can be expressed in terms of the modified Bessel functions:

$$F_1(z;s)=z^{-1/2}I_0(\zeta),\quad F_2(z;s)=z^{-1/2}K_0(\zeta), \quad (2.123)$$

where

$$\zeta=2sz^{-1}. \quad (2.124)$$

Case (iv): $c(x)$ for $\sigma\neq 0$

The corresponding characteristic equations are

$$\frac{dx}{2\alpha(x)\beta(t)}=\frac{\sigma^2dt}{2(\alpha'-H\alpha)\beta'(t)}=\frac{du}{[\alpha H\beta(t)+s]u}, \quad (2.125)$$

where

$$\beta(t)=pe^{\sigma t}+qe^{-\sigma t}, \quad (2.126)$$

and $\alpha(x)$, $H=c'/c$ satisfy Eqs. (2.9) and (2.21). The similarity variable is

$$z=(c/\alpha)(pe^{\sigma t}-qe^{-\sigma t}). \quad (2.127)$$

The corresponding form for the similarity solutions is

$$u=\sqrt{c}\left[\frac{w}{\beta\sqrt{K}+2\sqrt{pq(K+\sigma^2w^2)}}\right]^\rho F(z;s), \quad (2.128)$$

where

$$w=\alpha/c,\quad \rho=s/4\sqrt{pqK}. \quad (2.129)$$

$F(z;s)$ satisfies the ODE

$$(K^2z^2-4pq\sigma^2)F_{zz}(z;s)+2K(1-\rho)zF_z(z;s)$$
$$+\{\tfrac{1}{4}+\rho(\rho-1)K\}F(z;s)=0. \quad (2.130)$$

III. THE INVARIANCE PROPERTIES OF THE SYSTEM

Clearly (1.2) is always invariant under translations in t and uniform scalings of u and v.

If the point transformation

$$X=x+\epsilon\xi(x,t)+O(\epsilon^2),$$
$$T=t+\epsilon\tau(x,t)+O(\epsilon^2),$$
$$U=u+\epsilon[f(x,t)u+g(x,t)v]+O(\epsilon^2),$$
$$V=v+\epsilon[k(x,t)v+l(x,t)u]+O(\epsilon^2), \quad (3.1)$$

leaves invariant (1.2), then $\{\xi,\tau,f,g,k,l\}$ satisfy determining equations which reduce to

$$k_t - g_x = 0, \tag{3.2a}$$

$$l_t - f_x = 0, \tag{3.2b}$$

$$c^2(x)l - g = 0, \tag{3.2c}$$

$$c^2(x)\tau_x - \xi_t = 0, \tag{3.2d}$$

$$c^2(x)k_x - g_t = 0, \tag{3.2e}$$

$$c^2(x)l_x - f_t = 0, \tag{3.2f}$$

$$c(x)[\tau_t - \xi_x] + c'(x)\xi = 0, \tag{3.2g}$$

$$\xi_x - \tau_t + k - f = 0. \tag{3.2h}$$

The consistency of Eqs. (3.2b), (3.2c), and (3.2f) leads to $g(x,t)$ satisfying

$$g_x H + gH' = 0, \tag{3.3}$$

where $H = c'/c$. Then $g(x,t)$ satisfies

$$g(x,t) = -a(t)/2H. \tag{3.4}$$

Moreover if $a(t) \neq 0$, then it is necessary that $\{c(x), a(t)\}$ satisfy

$$cc'(c/c')'' = a''(t)/a(t) = \text{const} = \lambda^2. \tag{3.5}$$

If $a(t) = 0$, then either $c(x)$ solves (3.5) with $\lambda = 0$ or (1.2) is only invariant under above-mentioned scalings of u and v and translations in t.

In the following subsections we will show that system (1.2) is invariant under a four-parameter Lie group of point transformations of the form (3.1) if and only if $c(x)$ satisfies the ODE (3.5), namely,

$$cc'(c/c')'' = \lambda^2. \tag{3.6}$$

The general solution of (3.6) is derived in Appendix B. It turns out that if $\lambda \neq 0$, the general solution of (3.6) does not solve (2.9), (2.21). Note that λ can be real or imaginary. The case $\lambda = 0$ will be considered in the following subsection and the case $\lambda \neq 0$ in Sec. III B.

A. The case $\lambda = 0$

The general solution of

$$(c/c')'' = 0 \tag{3.7}$$

leads to the consideration of three separate subcases.

Case (i): $c(x) = (Ax + B)^C$, $C \neq 0,1$

The same subsitutions that reduced (2.41) to (2.42) lead here to the equivalent system

$$v_t = u_x, \quad u_t = x^{2C} v_x. \tag{3.8}$$

The solution of the determining equations (3.2a)–(3.2h) leads to

$$\begin{aligned} \xi &= px + 2qxt, \\ \tau &= p(1-C)t + q[(1-C)t^2 + x^{2-2C}/(1-C)] + r, \\ f &= q(2C-1)t + s, \\ g &= -qx, \\ k &= -pC - qt + s, \\ l &= -qx^{1-2C}, \end{aligned} \tag{3.9}$$

where p, q, r, and s are arbitrary constants.

1. Similarity solutions of (3.8) for $q=r=0$, $p=1$, s arbitrary

The corresponding characteristic equations are

$$\frac{dx}{x} = \frac{dt}{(1-C)t} = \frac{du}{su} = \frac{dv}{(s-C)v}. \tag{3.10}$$

Comparing (2.43) and (3.10), one sees that the similarity solutions for u are of the form (2.44), (2.45). The corresponding solutions for v are of the form

$$v = x^{s-C} G(z;s). \tag{3.11}$$

Substituting (2.45) and (3.11) into the system (3.8), one finds that

$$\begin{aligned} G_z(z;s) &= sF(z;s) + (C-1)zF_z(z;s), \\ (s-C)G(z;s) + (C-1)zG_z(z;s) &= F_z(z;s). \end{aligned} \tag{3.12}$$

If one eliminates $G(z;s)$ from (3.12), then $F(z;s)$ solves (2.46). Correspondingly

$$G(z;s) = \frac{[1-(C-1)^2z^2]F_z(z;s) + (1-C)szF(z;s)}{s-C}. \tag{3.13}$$

2. Similarity solutions of (3.8) for $p=r=0$, $q=1$, s arbitrary

First we find the global transformation (3.1) corresponding to (3.9) for $p = r = s = 0$, $q = 1$. Then it is easy to obtain the global transformation for arbitrary s. This global transformation leads to the similarity form of the solutions.

The global transformation for $p = r = s = 0$, $q = 1$, is found by solving the characteristic differential equations

$$\begin{aligned} \frac{dX}{2XT} &= \frac{dT}{(1-C)T^2 + X^{2-2C}/(1-C)} \\ &= \frac{dU}{(2C-1)TU - XV} \\ &= \frac{dV}{-[TV + X^{1-2C}U]} = d\epsilon, \end{aligned} \tag{3.14}$$

where $X = x$, $T = t$, $U = u$, $V = v$, at $\epsilon = 0$.

The first equality in (3.14) leads to

$$\begin{aligned} &(1-C)T^2X^{C-1} - \frac{1}{(1-C)X^{C-1}} \\ &= \text{const} = (1-C)t^2x^{C-1} - \frac{1}{(1-C)x^{C-1}} = z. \end{aligned} \tag{3.15}$$

Next we consider the differential equations

$$\frac{dU}{d\epsilon} = (2C-1)TU - XV; \tag{3.16}$$

$$\frac{dV}{d\epsilon} = -TV - X^{1-2C}U. \tag{3.17}$$

One can show that

$$\frac{d^2V}{d\epsilon^2} = -2CT\frac{dV}{d\epsilon} + \left[\frac{C}{C-1}X^{2-2C} - CT^2\right]V. \tag{3.18}$$

Let $V = X^{-C/2}W$. Then (3.18) reduces to

$$\frac{d^2W}{d\epsilon^2} = 0. \tag{3.19}$$

Hence

$$V=X^{-C/2}(F\epsilon+G), \tag{3.20}$$

where F and G are constants. Equation (3.17) leads to

$$U=X^{3C/2-1}[(C-1)T(F\epsilon+G)-F]. \tag{3.21}$$

The solution of

$$\frac{dX}{2XT}=d\epsilon \tag{3.22}$$

leads to

$$[1+z(1-C)X^{C-1}]^{1/2}=z(C-1)(\epsilon+E), \tag{3.23}$$

where E is a constant.

The global transformation for arbitrary s, $p=r=0$, $q=1$, follows:

$$\begin{aligned}&[1+z(1-C)X^{C-1}]^{1/2}=z(C-1)(\epsilon+E),\\&(1-C)T^2X^{C-1}-\frac{1}{(1-C)X^{C-1}}=z,\\&U=e^{s\epsilon}X^{3C/2-1}[(C-1)T(F\epsilon+G)-F],\\&V=e^{s\epsilon}X^{-C/2}(F\epsilon+G),\end{aligned} \tag{3.24}$$

where the constants $\{z,E,F,G\}$ can be expressed in terms of $\{x,t,u,v\}$ by solving (3.24) at $\epsilon=0$. The explicit form of the global transformation is easily found by solving (3.24) for $\{X,T,U,V\}$.

The corresponding similarity solutions are found by letting z play the role of the similarity variable, and letting $\{E,F,G\}$ be arbitrary functions of z and s. Without loss of generality one can set $E=0$. Solving the first two equations of (3.24) for ϵ, one then finds that the resulting similarity form is

$$\begin{aligned}u&=e^{stx^{C-1}z^{-1}}x^{3C/2-1}[(C-1)t\{x^{C-1}z^{-1}F(z;s)\\&\quad+G(z;s)\}-F(z;s)],\\v&=e^{stx^{C-1}z^{-1}}x^{-C/2}[tx^{C-1}z^{-1}F(z;s)+G(z;s)].\end{aligned} \tag{3.25}$$

If one substitutes (3.25) into the system (3.8) then $F(z;s)$ and $G(z;s)$ satisfy a corresponding system of coupled first-order linear ODE's.

Case (ii): $c(x)=x$

Here the system (1.2) becomes

$$v_t=u_x,\quad u_t=x^2v_x. \tag{3.26}$$

The solution of the determining equations (3.2a)–(3.2h) leads to

$$\begin{aligned}&\xi=px+2qxt,\quad \tau=2q\log x+r,\\&f=qt+s,\quad g=-qx,\\&k=-p-qt+s,\quad l=-qx^{-1}.\end{aligned} \tag{3.27}$$

3. Similarity solutions of (3.26) for $q=r=0$, $p=1$, s arbitrary

The resulting similarity solutions are easily found to be of the form

$$u=x^sF(t;s),\quad v=x^{s-1}G(t;s). \tag{3.28}$$

$F(t;s)$ is any solution of (2.59) and

$$G(t;s)=(s-1)^{-1}F_t(t;s). \tag{3.29}$$

4. Similarity solutions of (3.26) for $p=r=0$, $q=1$, s arbitrary

Here the same procedure is followed as in Case (i). The resulting global transformation can be written as

$$\begin{aligned}&T+\log X=Ee^{2\epsilon},\\&T^2-(\log X)^2=z,\\&U=e^{2\epsilon s}X^{1/2}(e^{2\epsilon}F+G),\\&V=e^{2\epsilon s}X^{-1/2}(G-e^{2\epsilon}F).\end{aligned} \tag{3.30}$$

The resulting similarity form is

$$\begin{aligned}u&=x^{1/2}|t+\log x|^s[|t+\log x|F(z;s)+G(z;s)],\\v&=x^{-1/2}|t+\log x|^s[G(z;s)-|t+\log x|F(z;s)],\end{aligned} \tag{3.31}$$

where $\{F(z;s),G(z;s)\}$ are to be determined by substitution of (3.31) into (3.26).

Case (iii): $c(x)=e^{-x/2}$

Here the system (1.2) is

$$v_t=u_x,\quad u_t=e^{-x}v_x. \tag{3.32}$$

The solution of the determining equations (3.2a)–(3.2h) leads to

$$\begin{aligned}&\xi=2p+4qt,\quad \tau=pt+q(t^2+4e^x)+r,\\&f=-2qt+2s,\quad g=-2q,\\&k=p+2s,\quad l=-2qe^x.\end{aligned} \tag{3.33}$$

5. Similarity solutions of (3.32) for $q=r=0$, $p=1$, s arbitrary

The similarity variable is

$$z=te^{-x/2}. \tag{3.34}$$

The form of the solutions is

$$u=e^{sx}F(z;s),\quad v=e^{(s+1/2)x}G(z;s). \tag{3.35}$$

$F(z;s)$ is any solution of (2.72) and

$$G(z;s)=(2s+1)^{-1}[(2-\tfrac{1}{2}z^2)F_z(z;s)+szF(z;s)]. \tag{3.36}$$

6. Similarity solutions of (3.32) for $p=r=0$, $q=1$, s arbitrary

The resulting global transformation (3.1) can be written as

$$\begin{aligned}&T^2e^{-x/2}-4e^{x/2}=z,\\&2z^{-1}\sqrt{4+ze^{-x/2}}=\epsilon+E,\\&U=e^{\epsilon s}e^{-3x/4}[F-\tfrac{1}{2}T(F\epsilon+G)],\\&V=e^{\epsilon s}e^{x/4}(F\epsilon+G).\end{aligned} \tag{3.37}$$

The resulting similarity form is

$$\begin{aligned}u&=-e^{2sz^{-1}te^{-x/2}}e^{-3x/4}[4z^{-1}e^{x/2}F(z;s)+\tfrac{1}{2}tG(z;s)],\\v&=e^{2sz^{-1}te^{-x/2}}e^{x/4}[2z^{-1}te^{-x/2}F(z;s)+G(z;s)],\end{aligned} \tag{3.38}$$

where $\{F(z;s),G(z;s)\}$ are determined by substitution of (3.38) into (3.32).

B. The case $\lambda \neq 0$

By appropriate scalings of c and x, Eq. (3.6) reduces to (see Appendix B)

$$c' = \nu^{-1} \sinh(\nu \log c) \tag{3.39}$$

or

$$c' = \nu^{-1} \sin(\nu \log c) \tag{3.40}$$

for $\lambda^2 > 0$. For $\lambda^2 < 0$, Eq. (3.6) reduces to

$$c' = \nu^{-1} \cosh(\nu \log c). \tag{3.41}$$

In Eqs. (3.39)–(3.41), ν is an arbitrary real constant. If $\nu = 1$, then $c(x) = \sqrt{1 + e^x}$ solves (3.39).

In the cases {(3.39), (3.40)}, the solution of the determining equations (3.2a)–(3.2h) leads to

$$\begin{aligned}
\xi &= (2c/c')[pe^t + qe^{-t}],\\
\tau &= 2[(c/c')' - 1][pe^t - qe^{-t}] + r,\\
f &= [2 - (c/c')'][pe^t + qe^{-t}] + s,\\
g &= -(c/c')[pe^t - qe^{-t}],\\
k &= -(c/c')'[pe^t + qe^{-t}] + s,\\
l &= -(1/cc')[pe^t - qe^{-t}].
\end{aligned} \tag{3.42}$$

The similarity solutions for wave speeds $c(x)$ satisfying (3.39), (3.40), or (3.41) will be constructed in a future paper.

IV. INVARIANCE PROPERTIES OF THE SINGLE EQUATION VIS-A-VIS THE SYSTEM WHEN $c(x) \neq$ const

The single equation (1.1) is invariant under a four-parameter Lie group of point transformations, $\{p,q,r,s\}$, if and only if $c(x)$ solves Eqs. (2.21) and (2.9). This corresponds to a five-parameter family for $c(x)$.

If

$$c = \Psi(x,\sigma,K) \tag{4.1}$$

is a solution of {(2.21), (2.9)}, it follows from their invariance properties that

$$c = k_1 \Psi(k_3 x + k_2, \sigma, K) \tag{4.2}$$

is the general solution of {(2.21), (2.9)}, where $\{k_1, k_2, k_3\}$ are arbitrary constants.

The system (1.2) is invariant under a four-parameter Lie group of point transformations if and only if $c(x)$ solves Eq. (3.6). This corresponds to a four-parameter family for $c(x)$. If

$$c = \Phi(x,\nu) \tag{4.3}$$

solves (3.39), (3.40), or (3.41) then it follows that

$$c = k_1 \Phi((k_1/\lambda)x + k_2, \nu) \tag{4.4}$$

is the general solution of (3.6) where $\{k_1, k_2, \nu\}$ are arbitrary constants.

One can show that the single equation (1.1) and the system of equations (1.2), for the same $c(x)$, admit a four-parameter Lie group of point transformation if and only if

$$c(x) = (A + Bx)^C, \tag{4.5}$$

or the limiting case

$$c(x) = Ae^{Bx}, \tag{4.6}$$

where $\{A, B, C\}$ are arbitrary constants. However, it could still follow that an invariant solution of (1.2) maps into a noninvariant solution of the wave equation (1.1) under the mapping (1.4). In fact if $c(x)$ is of the form (4.5) or (4.6), an invariant solution of (1.2) maps into an invariant solution of (1.1), under the mapping (1.4), if and only if the invariant solution of (1.2) has $q = 0$.

The group leaving invariant the single equation (1.1) is infinite if and only if

$$c(x) = (A + Bx)^2. \tag{4.7}$$

The group leaving invariant the system (1.2) contains at most four parameters.

Any Lie group of point transformations (3.1), leaving invariant (1.2), can be expressed in the equivalent form

$$\begin{aligned}
X &= x, \quad T = t,\\
U &= u + \epsilon\eta(x,t,u,v,u_x,u_t) + O(\epsilon^2),\\
V &= v + \epsilon\zeta(x,t,u,v,u_x,u_t) + O(\epsilon^2),
\end{aligned} \tag{4.8}$$

where

$$\eta = f(x,t)u + g(x,t)v - \xi(x,t)u_x - \tau(x,t)u_t, \tag{4.9}$$

$$\zeta = k(x,t)v + l(x,t)u - \tau(x,t)u_x - \xi(x,t)c^{-2}(x)u_t. \tag{4.10}$$

The symmetry (4.8) of the system (1.2) is the symmetry

$$X = x, \quad T = t, \quad U = u + \epsilon\tilde{\eta} + O(\epsilon^2), \tag{4.11}$$

of (1.1), where

$$\tilde{\eta} = \eta(x,t,u,D_t^{-1}u_x,u_x,u_t), \tag{4.12}$$

and D_t^{-1} is the operator inverse to the total derivative operator D_t defined by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots. \tag{4.13}$$

If η depends explicitly on v, i.e., $g \neq 0$ in (4.9), then accordingly $\tilde{\eta}$ depends explicitly on $D_t^{-1}u_x$ and consequently the resulting transformation is neither a Lie group of point transformations nor more generally a Lie–Bäcklund transformation.[5,6] If the group parameter $q \neq 0$ in (3.9), (3.27), (3.33), and (3.42), then $g \neq 0$. If η is independent of v, i.e., $g = 0$ in (4.9), then the symmetry (4.11) corresponds to a Lie group of point transformations admitted by the wave equation (1.1), and in this case the invariant (similarity) solutions of (1.2) map into invariant solutions of (1.1) under the mapping (1.4).

Conversely, let

$$\begin{aligned}
X &= x, \quad T = t,\\
U &= u + \epsilon\eta(x,t,u,u_x,u_t) + O(\epsilon^2),
\end{aligned} \tag{4.14}$$

be a Lie group of point transformations, equivalent to (2.1), leaving invariant (1.1). Then

$$\eta = f(x,t)u - \xi(x,t)u_x - \tau(x,t)u_t. \tag{4.15}$$

The corresponding symmetry of (1.2) is

$$\begin{aligned}
X &= x, \quad T = t, \quad U = u + \epsilon\eta + O(\epsilon^2),\\
V &= v + \epsilon\zeta + O(\epsilon^2),
\end{aligned} \tag{4.16}$$

where ζ satisfies the compatible system of PDE's

$$D_t\zeta = D_x\eta, \quad D_x\zeta = c^{-2}(x)D_t\eta, \tag{4.17}$$

and D_x is the total derivative operator

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots . \tag{4.18}$$

Although (4.17) always has a solution ζ for any η of the form (4.15), ζ cannot necessarily be expressed in terms of $\{x,t,u,v\}$ and the partial derivatives of u. If this is the case the symmetry (4.16) is not a Lie–Backlünd transformation.

V. EQUIVALENCE CLASSES OF THE SINGLE EQUATION

A natural question arises as to whether PDE's of the form (1.1) or (1.2), admitting a four-parameter Lie group of point transformations, are equivalent to each other in the sense that there exists a point transformation mapping one PDE into the other. Lie[1,2] gave a criterion applicable to the single PDE (1.1). When Eq. (1.1) is invariant under a four-parameter Lie group of point transformations, Lie's criterion reduces simply to the following statement.

Wave equations of the form (1.1) admitting a four-parameter group are equivalent if and only if the corresponding wave speeds c(x) have the same value for the integration constant K in Eq. (2.21).

For $\sigma = 0$ and any value of K, $-\infty < K < \infty$, there exists a solution $c(x)$ of system $\{(2.9), (2.21)\}$. As noted previously $\alpha(x)$ can be imaginary. Hence the wave speed $c(x)$ for any $\sigma \neq 0$ is equivalent to some wave speed $c(x)$ for $\sigma = 0$.

For $\sigma = 0$, the following wave speeds $c(x)$ are equivalent, modulo scalings in c and x and translations in x:

(a) $c(x) = x, x^2+1, x^2-1$;

(b) $c(x) = e^x, x^2e^{1/x}$;

(c) $c(x) = x^C, x^{2-C}, (1-x)^C(1+x)^{2-C}$, for any C.

VI. CONCLUSIONS

In this paper we have given the complete group classification of the wave equation (1.1) and the corresponding system (1.2). We have shown that for a wide class of wave speeds $c(x)$, (1.2) is invariant under a larger group than (1.1). Consequently for such wave speeds, whose canonical equations are (3.39), (3.40), and (3.41), there exist invariant (similarity) solutions of (1.2) which are noninvariant solutions of (1.1).

In a future paper we will discuss some interesting solutions of (1.1) for wave speeds $c(x)$ solving (3.39), (3.40), or (3.41). These include solutions for a class of wave speeds with the following physically significant properties:

(a) $c(x)$ is monotone on $(-\infty, \infty)$;

(b) $\lim_{x\to-\infty} c(x) = A, \quad \lim_{x\to+\infty} = B$;

(c) $\max_{x\in\mathbf{R}} |c'(x)| = C$;

(d) $c(0) = D$;

where $\{A,B,C,D\}$ are arbitrary positive constants, provided D is between A and B.

In another future paper we will shown how to use the invariance properties of the system (1.2) to linearize some nonlinear systems of PDE's which cannot be linearized by hodograph transformations, applying procedures outlined in Ref. 7.

APPENDIX A: THE GENERAL SOLUTIONS OF EQS. (2.9) and (2.21)

Here we find the general solution for $c(x)$ of the system

$$(\alpha' - H\alpha)^2 - \sigma^2\alpha^2/c^2 = K, \tag{A1}$$

$$\alpha^2 = (2H' + H^2)^{-1}, \tag{A2}$$

where $H = c'/c$, $\sigma \neq 0$. An integration of Eq. (2.20) to Eq. (A1) resulted from taking the commutator of L_p with L_q in (2.30). Without loss of generality, $\sigma = 1$, by an obvious scaling of c.

First we factor (A1) as

$$(\alpha' - H\alpha + \alpha/c)(\alpha' - H\alpha - \alpha/c) = K. \tag{A3}$$

Now let

$$h(x) = \alpha' - \alpha c'/c + \alpha/c. \tag{A4}$$

Then

$$\alpha' - \alpha(c'/c) - \alpha/c = K/h(x). \tag{A5}$$

Equations (A4) and (A5) lead to

$$c = h/h', \quad \alpha = \tfrac{1}{2}[(h^2 - K)/h']. \tag{A6}$$

Thus the problem of finding $c(x)$ is equivalent to finding $h(x)$ satisfying (A2) which now becomes

$$(h^2 - K)[2h'''h'h^2 - 3(h'')^2h^2 + (h')^4] = (h')^4h^2. \tag{A7}$$

Equation (A7) is invariant under arbitrary scalings and translations in x. Hence[3] one can reduce (A7) to a first-order ODE by choosing corresponding differential invariants

$$u = h, \quad v = h''/(h')^2. \tag{A8}$$

Then (A7) becomes the Riccati equation

$$2\frac{dv}{du} + v^2 + \frac{1}{u^2 - K} + \frac{1}{u^2} = 0. \tag{A9}$$

After v is solved explicitly in terms of u, $v = v(u)$, (A8) becomes

$$h''/h' = v(h)h'. \tag{A10}$$

Thus

$$\log h' = \int v(h)dh + k_1 = -\log M(h), \tag{A11}$$

where k_1 is an arbitrary constant. Then

$$\int M(h)dh = x + k_2, \tag{A12}$$

where k_2 is an arbitrary constant. After solving (A12) for $h(x)$, (A6) leads to

$$c(x) = h(x)M(h(x)). \tag{A13}$$

It should be noted that the transformation

$$v = 2w^{-1}\frac{dw}{du} \tag{A14}$$

reduces (A9) to the second-order linear ODE

$$4\frac{d^2w}{du^2} + \left(\frac{1}{u^2-k} + \frac{1}{u^2}\right)w = 0. \tag{A15}$$

Equation (A15) can be solved in terms of hypergeometric functions.

APPENDIX B: THE GENERAL SOLUTION OF EQ. (3.6)

Here we find the general solution of Eq. (3.6) when $\lambda \neq 0$. Without loss of generality, $\lambda = 1$ or i, by an appropriate scaling of $c(x)$. Hence we consider

$$cc'(c/c')'' = \pm 1. \tag{B1}$$

This ODE can be fully integrated using group methods described in Ref. 3.

Since (B1) is invariant under scalings $x^* = \mu x$, $c^* = \mu c$, and translations in x, we choose new variables[3]

$$u = c', \quad v = cc'', \tag{B2}$$

which are differential invariants with respect to this two-parameter family of symmetries. Consequently (B1) becomes

$$\frac{dv}{du} = \frac{2v}{u} \mp \frac{u}{v}. \tag{B3}$$

Equation (B3) is homogeneous in u and v. Using this fact, one finds that the general solution of (B3) is

$$u^{-2}[(v/u)^2 \mp 1] = \text{const} = \nu^2. \tag{B4}$$

Now we choose new variables c and u, invariants under translations in x, so that (B4) becomes

$$\frac{du}{dc} = \frac{\sqrt{\nu^2u^2 \pm 1}}{c}. \tag{B5}$$

The general solution of (B5) is then

$$u = (1/2\nu)[(\rho c)^\nu \mp (\rho c)^{-\nu}], \tag{B6}$$

where ρ is an arbitrary constant. After scaling c and x so that ρ becomes 1, (B1) reduces to

$$c' = (1/2\nu)[c^\nu \mp c^{-\nu}], \tag{B7}$$

i.e.,

$$c' = \nu^{-1}\sinh(\nu \log c) \quad \text{or} \quad \nu^{-1}\cosh(\nu \log c). \tag{B8}$$

If ν^2 is replaced by $-\nu^2$ in (B4) then (B7) reduces to

$$c' = \nu^{-1}\sin(\nu \log c). \tag{B9}$$

Equation (B8) can be integrated out if ν is any rational number.

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