Journal of Nonlinear Mathematical Physics Volume 15, Supplement 1 (2008), 1–24 Article

Nonlocal Extensions of Similarity Methods

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Abstract

Similarity methods include the calculation and use of symmetries and conservation laws for a given partial differential equation (PDE). There exists a variety of software to calculate and use local symmetries and local conservation laws. However, it is often the case that a given PDE admits no useful local symmetry or local conservation law. It is shown how to construct systematically trees of nonlocally related but equivalent systems of PDEs. A local symmetry or local conservation law of a PDE in such a tree could yield a useful nonlocal symmetry or nonlocal conservation law for a given PDE within such a tree. If this is the case, one can extend the usefulness of similarity methods for a given PDE. Many examples are given.

1 Introduction

The aim of similarity methods is to systematically find solutions and/or conservation laws for nonlinear partial differential equations (PDEs). Similarity methods are related to symmetries admitted by PDEs. In general, a *symmetry* admitted by a system of PDEs is any transformation that leaves invariant its solution space, i.e., each solution surface of the system is mapped to another solution surface of the same system under an admitted symmetry. Admitted continuous symmetries correspond to deformations of the surfaces of the system's solution space-a topological characterization. Such a characterization of symmetries is not coordinate-dependent and, in particular, not limited to invertible transformations of the coordinates (dependent and independent variables) of a given system of PDEs. Hence, in principle, any system of PDEs has symmetries. The problem is how to use and how to find (calculate) symmetries. Use and calculation of symmetries should go hand-in-hand to be of value.

In the late 19$^{\text{th}}$ century, the Norwegian mathematician Sophus Lie gave an algorithm to find point transformations (more generally contact transformations), depending on a continuous parameter, admitted by a given PDE. Lie's symmetry transformations are invertible and act on the given variables of the PDE. Their computation depends on solving a set of *determining equations* that are overdetermined linear systems of PDEs. Lie (1881) showed how to use such admitted *point* (*contact*) *symmetries* to find corresponding *invariant solutions* (*similarity solutions*) of the given PDE. Such special solutions are invariant, i.e., they map into themselves under the action of the admitted continuous symmetry. In particular, invariant solutions satisfy an *invariant surface condition* that naturally arises from their being mapped into themselves.

The American engineering scientist Buckingham (1914) introduced a systematic procedure to make every real equation *dimensionless*. Consequently, a boundary value problem (BVP) for a PDE system might be reduced to a BVP for a DE with fewer independent variables. If this is the case, one can show that the reduced solution arising from dimensional analysis is an invariant solution arising from admitted Lie point symmetries that are scalings of the variables (independent and dependent) of the given PDE system. From this point of view, Lie's work on finding invariant solutions for systems of PDEs is an extension of Buckingham's famous Pi-theorem applied to BVPs for systems of PDEs.

Lie's "classical" method for finding invariant solutions from admitted point symmetries can be extended to the nonclassical method [Bluman (1967), Bluman and Cole (1969), Levi and Winternitz (1989), Clarkson and Mansfield (1994)]. Here one finds solutions invariant under "symmetries" admitted by the augmented system consisting of the given system and the invariant surface condition-in effect one seeks solutions of the given system with the invariant surface condition as a side condition.

From knowledge of the admitted point (contact) symmetries for a given system of PDEs, one can determine whether or not it can be mapped invertibly into a PDE system that belongs to a target class of PDEs that can be completely characterized in terms of admitted point (contact) symmetries. In particular, from this point of view, one can determine whether or not a given nonlinear PDE can be mapped invertibly to a linear PDE and find such a mapping when it exists [Kumei and Bluman (1982), Bluman and Kumei (1990a)]; one can determine whether or not a linear PDE with variable coefficients can be mapped invertibly to a linear PDE with constant coefficients and find such a mapping when it exists [Bluman (1980, 1983), Bluman and Kumei (1990a)].

Lie's point and contact symmetries can be generalised to local (higher-order) symmetries admitted by a given PDE [Anderson, Kumei and Wulfman (1972), Olver (1977)]. Here a local symmetry is a solution of the linearised system (*Frêchet derivative*) of the given PDE that holds for *all* solutions of the given PDE. Here one seeks solutions of the linearised system that depend on derivatives to a specific order of the solutions of the given system. Point symmetries correspond to solutions of the linearised system that depend at most linearly on first derivatives; contact symmetries correspond to any such solutions that depend at most on first derivatives; higher-order symmetries correspond to any such solutions that depend at least on second derivatives.

Emmy Noether (1918) showed that if a system of PDEs admits a variational principle (variational system), then a local symmetry leaving invariant the action integral for its Lagrangian density (*variational symmetry*) yields a *conservation law*. Conversely, for a given variational system, all local conservation laws arise from variational symmetries. Hence there is a direct one-to-one correspondence between conservation laws and admitted variational symmetries of a variational system. Moreover, one can show that a variational symmetry must be a local symmetry admitted by the variational system; the converse does not hold in general, i.e., there do exist local symmetries of variational systems that are not variational symmetries.

Conservation laws can be found systematically and directly for any system of PDEs, even if is not variational [Anco and Bluman (1997, 2002), Wolf (2002a)]. When a system is variational, then the *multipliers* (factors, characteristics) yielding conservation laws must be symmetries. In particular, here multipliers solve a linear determining system of PDEs that includes the linearised system of the given system of PDEs. When a given system of PDEs is not variational, the multipliers solve a linear determining system of PDEs that includes the *adjoint* of the linearised system of the given system of PDEs [It is well-known that a given system of PDEs, as written, admits a variational principle if and only if its linearised system is self-adjoint [Olver (1986)]]. From knowledge of its admitted conservation law multipliers, one can also determine whether or not a given nonlinear system of PDEs can be mapped invertibly to a linear system [Bluman and Doran-Wu (1995)] and find an explicit mapping when such a mapping exists [Anco, Bluman and Wolf (2008)].

There is a variety of software available to obtain the local symmetries and local conservation laws for a given system of differential equations [Hereman (1997), Wolf (2002b), Cheviakov (2007)].

A severe limitation in using similarity methods is that a given system of PDEs as it stands does not admit a useful local symmetry or useful local conservation law. Hence in order to extend similarity methods, one needs to find nonlocally related but equivalent systems of PDEs for a given system of PDEs. In particular, there is not an invertible mapping relating such systems but each solution of a nonlocally related system yields a solution of the given system and each solution of the given system yields a solution of the nonlocally related system. But the relationship between solutions is not one-to-one.

A natural way to find nonlocally related systems begins with the use of admitted local conservation laws [Bluman and Kumei (1987), Bluman, Kumei and Reid (1988)]. Each conservation law yields a potential variable and a corresponding nonlocally related potential system. Moreover, n local conservation laws can yield $2^n - 1$ nonlocally related systems through combinations of the corresponding potential systems [Bluman and Cheviakov (2005), Bluman, Cheviakov and Ivanova (2006)]. Furthermore, other nonlocally related systems can arise as subsystems of any given system through elimination of one or more dependent variables. Nonlocally related subsystems can also arise following an interchange of independent and independent variables. Consequently, for a given system of PDEs one can obtain trees of nonlocally related but equivalent systems arising from n local conservation laws. Point symmetries (local conservation laws) of any of these nonlocally related systems could yield nonlocal symmetries (nonlocal conservation laws) of other systems in a tree, including the given system. In turn, such nonlocal symmetries could yield further invariant solutions, further useful mappings to simpler systems [Bluman and Kumei (1990b), Bluman and Shtelen (1996, 2004)] and further extensions of the nonclassical method to obtain additional specific solutions [Bluman and Yan (2005)].

2 Local symmetries admitted by PDEs and applications

Consider the situation of one dependent variable u and two independent variables (x, t). A one-parameter (ε) Lie group of point transformations (*point symmetry*) acting on a space of two independent variables (x, t) and dependent variable u is of the form

$$
\begin{aligned}
x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2) = e^{\varepsilon X}x = X(x,t,u;\varepsilon)\ ,\\
t^* &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2) = e^{\varepsilon X}t = T(x,t,u;\varepsilon)\ ,\\
u^* &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2) = e^{\varepsilon X}u = U(x,t,u;\varepsilon)\ ,
\end{aligned} \tag{2.1}
$$

in terms of its *infinitesimal generator*

$$
X = \xi(x,t,u)\,\frac{\partial}{\partial x} + \tau(x,t,u)\,\frac{\partial}{\partial t} + \eta(x,t,u)\,\frac{\partial}{\partial u}\ . \tag{2.2}
$$

The point symmetry (2.1) can also be found from its infinitesimal generator (2.2) by solving a corresponding initial value problem for an autonomous system of first order ODEs given by

$$
\begin{aligned}
\frac{dx^*}{d\varepsilon} &= \xi(x^*,t^*,u^*)\ ,\\
\frac{dt^*}{d\varepsilon} &= \tau(x^*,t^*,u^*)\ ,\\
\frac{du^*}{d\varepsilon} &= \eta(x^*,t^*,u^*)\ ,
\end{aligned} \tag{2.3}
$$

with $u^* = u$, $x^* = x$, $t^* = t$ when $\varepsilon = 0$.

A point symmetry (2.1) leaves invariant contact conditions relating differentials and through this naturally extends to its actions on derivatives through the extensions (prolongations)

$$
\begin{aligned}
(u_x)^* &= u_x + \varepsilon\eta^x(x,t,u,u_x,u_t) + O(\varepsilon^2) = e^{\varepsilon X^{(1)}}u_x\ ,\\
(u_t)^* &= u_t + \varepsilon\eta^t(x,t,u,u_x,u_t) + O(\varepsilon^2) = e^{\varepsilon X^{(1)}}u_t\ ,\\
(u_{xx})^* &= u_{xx} + \varepsilon\eta^{xx}(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) + O(\varepsilon^2) = e^{\varepsilon X^{(2)}}u_{xx}\ \text{, etc.,}
\end{aligned} \tag{2.4}
$$

where

$$
\begin{aligned}
X^{(1)} &= X + \eta^x\frac{\partial}{\partial u_x} + \eta^t\,\frac{\partial}{\partial u_t}\ ,\\
X^{(2)} &= X^{(1)} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\,\frac{\partial}{\partial u_{xt}} + \eta^{xx}\frac{\partial}{\partial u_{xx}}\ \text{, etc.,}
\end{aligned} \tag{2.5}
$$

with

$$
\begin{aligned}
\eta^x &= D_x\eta - (D_x\xi)u_x - (D_x\tau)u_t = \frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right)u_x - \frac{\partial\tau}{\partial x}\,u_t - \frac{\partial\xi}{\partial u}\,u_x^2 - \frac{\partial\tau}{\partial u}u_xu_t\ ,\\
\eta^t &= D_t\eta - (D_t\tau)u_t - (D_t\xi)u_x = \frac{\partial\eta}{\partial t} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right)u_t - \frac{\partial\xi}{\partial t}\,u_x - \frac{\partial\tau}{\partial u}\,u_t^2 - \frac{\partial\xi}{\partial u}\,u_xu_t\ ,\\
\eta^{xx} &= D_x\eta^x - (D_x\xi)u_{xx} - (D_x\tau)u_{tx} = \frac{\partial^2\eta}{\partial x^2} + \left(2\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2}\right)u_x - \frac{\partial^2\tau}{\partial x^2}\,u_t\\
&\quad + \left(\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi}{\partial x\partial u}\right)u_x^2 - 2\frac{\partial^2\tau}{\partial x\partial u}\,u_xu_t - \frac{\partial^2\xi}{\partial u^2}\,u_x^3 - \frac{\partial^2\tau}{\partial u^2}\,u_x^2u_t + \left(\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi}{\partial x}\right)u_{xx}\\
&\quad - 2\frac{\partial\tau}{\partial x}\,u_{xt} - 3\frac{\partial\xi}{\partial u}\,u_{xx}u_x - \frac{\partial\tau}{\partial u}\,u_{xx}u_t - 2\frac{\partial\tau}{\partial u}\,u_{xt}u_x\ \text{, etc.,}
\end{aligned} \tag{2.6}
$$

in terms of total derivative operators

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots \ , \ D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \cdots \ . \quad (2.7)$$

Consequently, one is able to find the set of infinitesimal generators admitted by a given system of PDEs. For further details, see any of the books by Ovsiannikov (1962, 1982), Bluman and Cole (1974), Olver (1986), Bluman and Kumei (1989), Stephani (1989), Hydon (2000), Bluman and Anco (2002) and Cantwell (2002).

As an example, consider the heat equation

$$u_{xx} = u_t \ . \quad (2.8)$$

One can show that the point symmetry (2.1) is admitted by the heat equation (2.8) if and only if its second extension $X^{(2)}$ satisfies

$$\begin{aligned} &\Big[e^{\varepsilon X^{(2)}}(u_{xx} - u_t)\Big]\Big|_{u_{xx}=u_t} = 0 \Leftrightarrow \Big[\eta^{xx} - \eta^t\Big]\Big|_{u_{xx}=u_t} = 0 \Leftrightarrow \\ &X = \xi(x,t)\,\frac{\partial}{\partial x} + \tau(t)\,\frac{\partial}{\partial t} + \Big[f(x,t)u + g(x,t)\Big]\,\frac{\partial}{\partial u} \end{aligned} \quad (2.9)$$

with

$$\begin{aligned} &\tau'(t) - 2\xi_x = 0 \ , \\ &2f_x - \xi_{xx} + \xi = 0 \ , \\ &f_{xx} - f_t = 0 \ , \\ &g_{xx} - g_t = 0 \ . \end{aligned} \quad (2.10)$$

After solving the linear determining system (2.10), one finds that the heat equation (2.8) admits six nontrivial point symmetries given by the infinitesimal generators

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x} \ , \qquad X_2 = \frac{\partial}{\partial t} \ , \qquad X_3 = x\,\frac{\partial}{\partial x} + 2t\,\frac{\partial}{\partial t} \ , \\ &X_4 = xt\,\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{1}{4}\,x^2 + \frac{1}{2}\,t\right)\,u\,\frac{\partial}{\partial u} \ , \\ &X_5 = t\,\frac{\partial}{\partial x} - \frac{1}{2}\,xu\,\frac{\partial}{\partial u} \ , \qquad X_6 = u\,\frac{\partial}{\partial u} \ . \end{aligned} \quad (2.11)$$

Under an admitted point symmetry X, *any solution* $u = \theta(x,t)$ of the heat equation (if not invariant) maps into the one-parameter family of solutions $u = \phi(x,t;\varepsilon)$ satisfying the functional equation

$$u = U\Big(e^{\varepsilon X}x, e^{\varepsilon X}t, \theta(e^{\varepsilon X}x, e^{\varepsilon X}t,); -\varepsilon\Big) \ .$$

An *invariant* (*similarity*) *solution* $u = \theta(x,t)$ satisfies

$$\phi(x,t;\varepsilon) = \theta(x,t) = U(e^{\varepsilon X}x, e^{\varepsilon X}t, \theta\left(e^{\varepsilon X}x, e^{\varepsilon X}t); -\varepsilon\right) \ . \quad (2.12)$$

One can show that $u = \theta(x,t)$ satisfies the functional equation (2.12) if and only if $u = \theta(x,t)$ satisfies the *invariant surface condition*

$$\xi u_x + \tau u_t = \eta \, .$$

From (2.9), we see that in the case of the nontrivial point symmetries ($g \equiv 0$) admitted by the heat equation (2.1), the invariant surface condition simplifies to the form

$$\xi(x,t)u_x + \tau(t)u_t = f(x,t)u \, .$$

Now consider the admitted point symmetry represented by the infinitesimal generator

$$X_4 = xt\,\frac{\partial}{\partial x} + t^2\,\frac{\partial}{\partial t} - \left(\frac{1}{4}\,x^2 + \frac{1}{2}\,t\right)\,u\,\frac{\partial}{\partial u} \, .$$

The corresponding characteristic equations (2.3) are given by

$$\begin{aligned} \frac{dx^*}{d\varepsilon} &= x^* t^* \, , \\ \frac{dt^*}{d\varepsilon} &= (t^*)^2 \, , \\ \frac{du^*}{d\varepsilon} &= -\left[\frac{1}{4}(x^*)^2 + \frac{1}{2}\,t^*\right]\,u^* \, , \end{aligned} \tag{2.13}$$

with $u^* = u$, $x^* = x$, $t^* = t$ when $\varepsilon = 0$. Solving (2.13), we obtain the corresponding one-parameter Lie group of point transformations

$$\begin{aligned} x* &= \frac{x}{1-\varepsilon t} \, , \\ t^* &= \frac{t}{1-\varepsilon t} \, , \\ u^* &= \sqrt{1-\varepsilon t}\;\exp\left[-\frac{\varepsilon x^2}{4(1-\varepsilon t)}\right]\,u \, , \end{aligned} \tag{2.14}$$

admitted by the heat equation (2.8). The group of transformations (2.14) maps any solution $u = \theta(x,t)$ of the heat equation (2.8), that is not invariant under X_4, into the one parameter family of solutions of the heat equation given by the expression

$$u = \phi(x,t;\varepsilon) = \frac{1}{\sqrt{1-\varepsilon t}}\;\exp\left[\frac{\varepsilon x^2}{4(1-\varepsilon t)}\right]\,\theta\left(\frac{x}{1-\varepsilon t}\,,\,\frac{t}{1-\varepsilon t}\right) \, .$$

The invariant solutions $u = \theta(x,t)$ arising from (2.14) satisfy the invariant surface condition (characteristic PDE)

$$xtu_x + t^2 u_t = -\left[\frac{1}{4}\,x^2 + \frac{1}{2}\,t\right]\,u \, . \tag{2.15}$$

Using the method of characteristics to solve (2.15), we first solve the first order characteristic ODEs

$$\frac{dx}{xt} = \frac{dt}{t^2} = \frac{du}{-\left[\frac{1}{4}\,x^2 + \frac{1}{2}\,t\right]\,u} \, . \tag{2.16}$$

The solution of the first ODE in (2.16) yields the *similarity variable* $\zeta(x,t) = \text{const}$. Then the solution of the second ODE in (2.16) yields the similarity form

$$u = \frac{1}{\sqrt{t}}\ \exp\left(-\frac{x^2}{4t}\right)\ F(\zeta) \tag{2.17}$$

for the corresponding invariant solution. Substitution of (2.17) into the heat equation yields the reduced ODE

$$F''(\zeta) = 0\ .$$

Consequently, we obtain the similarity solutions of the heat equation (2.1):

$$u = \left(C_1 + C_2\,\frac{x}{t}\right)\ \frac{1}{\sqrt{t}}\ \exp\left(-\frac{x^2}{4t}\right)\ ,$$

in terms of arbitrary constants C_1 and C_2.

2.1 Nonclassical method to find solutions of PDEs

Consider a PDE given by

$$u_{xx} = G(x,t,u,u_x,u_t,u_{tt},u_{xt})\ . \tag{2.18}$$

The *classical method* (Lie's reduction method) to obtain invariant solutions (*classical solutions*) of PDE (2.18) first involves finding an admitted point symmetry (infinitesimal generator) (2.2) so that

$$\left[X^{(2)}(u_{xx} - G)\right]\Big|_{u_{xx}=G} = 0\ ,$$

i.e., invariance for *every* solution of PDE (2.18). The resulting *linear* system of determining equations yields $\{\xi,\tau,\eta\}$, holding for *every* solution of PDE. Then one seeks invariant solutions $u = \theta(x,t)$ satisfying the corresponding invariant surface condition and the given PDE (2.18), i.e., $u = \theta(x,t)$ satisfying the constrained system

$$\begin{aligned} &\xi u_x + \tau u_t = \eta\ ,\\ &u_{xx} = G(x,t,u,u_x,u_t,u_{tt},u_{xt})\ . \end{aligned} \tag{2.19}$$

The *nonclassical method* to obtain solutions of PDE (2.18) first involves finding an infinitesimal generator

$$X = \xi(x,t,u)\,\frac{\partial}{\partial x} + \tau(x,t,u,)\,\frac{\partial}{\partial x} + \eta(x,t,u)\,\frac{\partial}{\partial u}$$

so that

$$\left[X^{(2)}(u_{xx} - G)\right]\Big|_{\substack{u_{xx}=G\\ \xi u_x+\tau u_t=\eta}} = 0\ , \tag{2.20}$$

i.e., invariance *only* for solutions of PDE (2.18) that simultaneously satisfy the system $u_{xx} = G$, $\xi u_x + \tau u_t = \eta$. Hence, without loss of generality, $\tau = 0$ or 1. The solution of the resulting *nonlinear* system of determining equations (2.20) yields $\{\xi, \eta\}$ holding only for *particular* solutions of PDE (2.18) that satisfy the constraining invariant surface condition. For the case $\tau = 1$, one then seeks for each such set of $\{\xi, \eta\}$, *nonclassical solutions* satisfying the corresponding system

$$\begin{aligned} &\xi u_x + u_t = \eta \; , \\ &u_{xx} = G(x, t, u, u_x, u_t, u_{tt}, u_{xt}) \; . \end{aligned} \tag{2.21}$$

It is easy to see that the nonclassical method yields *all solutions* of the given PDE (2.18) of the form

$$u = \Theta\Big(x, t, F\big(\zeta(x, t, u)\big)\Big)$$

with $F\big(\zeta(x, t, u)\big)$ satisfying a reduced ODE. Moreover, it is obvious that the nonclassical solutions of a PDE include all of its classical invariant solutions. Furthermore, any solution of a PDE obtained by the *direct method* of Clarkson and Kruskal (1989) is a nonclassical solution but, conversely, there exist nonclassical solutions that cannot be obtained by the direct method [Nucci and Clarkson (1992)]. For various nonlinear PDEs, many new nonclassical solutions have been found that are not obtainable as invariant solutions arising from reduction under point symmetry invariance.

2.2 Local symmetries

Given a PDE

$$G(x, t, u, \partial u, \ldots, \partial^k u) = 0 \; , \tag{2.22}$$

its admitted *local symmetries of order* p given by

$$\eta(x, t, u, \partial u, \ldots, \partial^p u) \, \frac{\partial}{\partial u} \tag{2.23}$$

are the solutions $\eta(x, t, u, \partial u, \ldots, \partial^p u)$ of its linearised system (*Frêchet derivative*)

$$\left[\frac{\partial G}{\partial u} \eta + \frac{\partial G}{\partial u_x} D_x \eta + \frac{\partial}{\partial u_t} D_t \eta + \frac{\partial^2 G}{\partial u_x^2} \, (D_x)^2 \eta + \cdots \right]_{\substack{G=0\,, \\ D_x G=0\,, \\ D_t G=0\,, \\ \vdots}} = 0;$$

$\partial^m u$ denotes the m^{th} partial derivatives of u. If $p = 1$, then a first order symmetry (2.23) is equivalent to a *contact symmetry*

$$\begin{aligned} &x^* = x + \varepsilon \frac{\partial \eta}{\partial u_x} + O(\varepsilon^2) \, , \; t^* = t + \varepsilon \frac{\partial \eta}{\partial u_t} + \cdots \, , \\ &u^* = u + \varepsilon \left(u_x \frac{\partial \eta}{\partial u_x} + u_t \frac{\partial \eta}{\partial u_t} - \eta \right) + \cdots \, , \\ &u_x^* = u_x + \varepsilon \left(-u_x \frac{\partial \eta}{\partial u} - \frac{\partial \eta}{\partial x} \right) + \cdots \, , \\ &u_t^* = u_t + \varepsilon \left(-u_t \frac{\partial \eta}{\partial u} - \frac{\partial \eta}{\partial t} \right) + \cdots \, . \end{aligned} \tag{2.24}$$

Moreover, if a first order symmetry (2.23) is linear in its first derivatives, i.e., it has an infinitesimal of the form

$$\eta(x,t,u\partial u) = \xi(x,t,u)u_x + \tau(x,t,u)u_t - \omega(x,t,u)$$

then it is equivalent to a *point symmetry*

$$x^* = x + \varepsilon\xi(u,t,u) + O(\varepsilon^2)\,, \quad t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2)\,,$$
$$u^* = u + \varepsilon\omega(x,t,u) + O(\varepsilon^2)\,.$$

In the characteristic (evolutionary) form (2.23), the action of a symmetry on the manifold of solution surfaces is over a fixed domain, i.e., independent variables are invariant. Here

$$x^* = x\,, \quad t^* = t\,, \quad u^* = u + \varepsilon\eta(x,t,u,\partial u,\ldots,\partial^p u) + O(\varepsilon^2)\,,$$

and a corresponding invariant solution $u = \theta(x,t)$ satisfies the constrained system

$$\eta(x,t,u,\partial u,\ldots,\partial^p u) = 0\,,$$
$$G(x,t,u,\partial u,\ldots,\partial^k u) = 0\,,$$

In terms of point symmetries written in the form (2.23), the heat equation (2.8) admits

$$X_1 = u_x\frac{\partial}{\partial u}\,, \qquad X_2 = u_t\frac{\partial}{\partial u}\,, \qquad X_3 = (xu_x + 2tu_t)\frac{\partial}{\partial u}\,,$$
$$X_4 = \left(xtu_x + t^2u_t + \left[\frac{1}{4}x^2 + \frac{1}{2}t\right]\right)\frac{\partial}{\partial u}\,,$$
$$X_5 = \left(tu_x + \frac{1}{2}xu\right)\frac{\partial}{\partial u}\,, \qquad X_6 = u\frac{\partial}{\partial u}\,.$$

As a second example, consider the Korteweg-de Vries (KdV) equation

$$u_{xxx} + uu_x + u_t = 0\,. \tag{2.25}$$

Let $\partial_x^k u = \frac{\partial^k u}{\partial x^k}$. In terms of the recursion operator

$$\mathbf{R} = (D_x)^2 + \frac{2}{3}u + \frac{1}{3}u_x(D_x)^{-1}\,,$$

one can show [Olver (1977)] that the KdV equation (2.25) admits an infinite sequence of local higher-order symmetries given by

$$\eta^n(u,\partial,u\ldots,\partial_x^{2n-1}u) = (\mathbf{R}^n)u_x\,, \quad n = 0,1,2,\ldots$$
$$\leftrightarrow u_x\frac{\partial}{\partial u}\,, \quad (uu_x + u_{xxx})\frac{\partial}{\partial u}\,, \quad \left(\frac{5}{6}u^2u_x + 4u_xu_{xx} + \frac{5}{3}uu_{xxx} + u_{xxxxx}\right)\frac{\partial}{\partial u}\,,\ldots\,.$$

Higher-order symmetries admitted by a PDE can be used to determine specific invariant solutions, a one-parameter family of solutions from any known solution that is not an invariant solution, in some cases determine whether or not a given PDE can be linearised [Bluman and Kumei (1980), Mikhailov, Shabat and Sokolov (1991) and references therein] and to find further nonclassical solutions through extending the nonclassical method to incorporate ansatzes related to higher-order symmetries [Fokas and Liu (1994)].

2.3 Local conservation laws

Given a PDE (2.22), a *local conservation law* is a divergence expression

$$D_x X(x,t,u,\partial u,\ldots,\partial^r u) + D_t T(x,t,u,\partial u,\ldots,\partial^r u) = 0, \tag{2.26}$$

holding for any solution of PDE (2.22). The search for nontrivial local conservation laws of PDE (2.22) is equivalent to finding all *local multipliers* $\mu(x,t,U,\partial U,\ldots,\partial^q U)$ so that for *arbitrary* differentiable functions U, one has the identity

$$\begin{aligned}&\mu(x,t,U,\partial U,\ldots,\partial^q U)G(x,t,U,\partial U,\ldots,\partial^k U)\\ &\equiv D_x X(x,t,U,\partial U,\ldots,\partial^r U) + D_t T(x,t,U,\partial U,\ldots,\partial^r U)\end{aligned} \tag{2.27}$$

holding for some conserved densities $\{X(x,t,U,\partial U,\ldots,\partial^r U),\ T(x,t,U,\partial U,\ldots,\partial^r U)\}$. One can show that $\mu(x,t,U,\partial U,\ldots,\partial^q U)$ is a multiplier for a local conservation law of PDE (2.22) if and only if

$$E_U\big(\mu(x,t,U,\partial U,\ldots,\partial^q U)G(x,t,U,\partial U,\ldots,\partial^k U)\big) \equiv 0 \tag{2.28}$$

in terms of the *Euler operator*

$$E_U = \frac{\partial}{\partial U} - \left(D_x\frac{\partial}{\partial U_x} + D_t\frac{\partial}{\partial U_t}\right) + \left((D_x)^2\frac{\partial}{\partial U_{xx}} + (D_t)^2\frac{\partial}{\partial U_n} + D_xD_t\frac{\partial}{\partial U_{xt}}\right) - \cdots$$

If the multipliers yielding a conservation law are known, it is relatively straightforward to obtain the corresponding conserved densities. In particular, there is an integral formula to obtain the conserved densities [Anco and Bluman (1997, 2002)]. Often it is easier to obtain the conserved densities by direct calculation through comparing like derivative terms in expression (2.27). For a comprehensive discussion of various approaches to obtain conserved densities, see Wolf (2002a).

2.4 Connections between local symmetries and local conservation laws

A multiplier $\mu(x,t,U,\partial U,\ldots,\partial^q U)$ for a conservation law of PDE (2.22) also yields a local symmetry

$$U = \mu(x,t,u,\partial u,\ldots,\partial^q u)\frac{\partial}{\partial u}$$

of PDE (2.22) if and only if PDE (2.22) admits a variational principle. In particular, when PDE (2.22), as written, admits a variational principle, then all solutions of the determining system (2.28) for its conservation law multipliers are local symmetries; but conversely, there can exist local symmetries of PDE (2.22) that are not variational symmetries.

As mentioned previously, U is a symmetry of PDE (2.22) if and only if $\mu(x,t,u,\partial u,\ldots,\partial^q u)$ is a solution of its linearized system (Frêchet derivative)

$$\left[\frac{\partial G}{\partial u}\mu + \frac{\partial G}{\partial u_x}D_x\mu + \frac{\partial G}{\partial u_t}D_t\mu + \frac{\partial^2 G}{\partial u_x^2}(D_x)^2\mu + \cdots\right]_{\substack{G=0\,,\\ D_xG=0\,,\\ D_tG=0\,,\\ \vdots}} = 0\,. \tag{2.29}$$

In particular, here the set of linear determining equations for local symmetries arising from (2.29) is a subset of the set of linear determining equations for multipliers arising from (2.28).

More generally, a multiplier $\mu(x, t, U, \partial U, \ldots, \partial^q U)$ for a conservation law of PDE (2.22) corresponds to a solution of the *adjoint system* of its linearized system given by (2.29). In particular, $\mu(x, t, u, \partial u, \ldots, \partial^q u)$ is a solution of the adjoint linear system

$$\left[\frac{\partial G}{\partial u}\mu - D_x\left(\frac{\partial G}{\partial u_x}\mu\right) - D_t\left(\frac{\partial G}{\partial u_t}\mu\right) + (D_x)^2\left(\frac{\partial^2 G}{\partial u_x^2}\mu\right) + \cdots\right]_{\substack{G=0\,,\\ D_xG=0\,,\\ D_tG=0\,,\\ \vdots}} = 0\,. \quad (2.30)$$

The linear system (2.30) is a subset of the set of linear determining equations for multipliers arising from (2.28). See Anco and Bluman (1997) for details.

Note that linear systems (2.29) and (2.30) are identical if and only if the linear operator defined by (2.29) is self-adjoint, i.e., the PDE (2.22) admits a variational principle.

Other important connections between local symmetries and local conservation laws include the following.

An invertible mapping from a given PDE to another PDE has the property that it maps a local conservation law of the given PDE to a local conservation law of the image PDE. Consequently, if the invertible mapping is a point or contact transformation (which could be a finite transformation) admitted by the given PDE, then it may yield one or more new local conservation laws of the given PDE from a known local conservation law [Bluman, Temuerchaolu and Anco (2006)].

For a given PDE, either its admitted point/contact symmetries or its admitted conservation law multipliers can be used to determine whether or not the PDE can be invertibly mapped to a linear PDE and to find such an explicit mapping when one exists [Anco, Bluman and Wolf (2008)].

3 Nonlocal extensions

A severe limitation in using similarity methods is that a given system of PDEs as it stands does not admit a useful local symmetry or local conservation law. But in principle, from the topological point of view, any given system of PDEs admits continuous symmetries.

Continuous symmetries that are not local symmetries are called *nonlocal symmetries*. The problem is to construct and utilise nonlocal symmetries. In particular, nonlocal symmetries have infinitesimal generators whose infinitesimals do not depend at most on a finite number of derivatives of the dependent variables of a given system.

Similarly, *nonlocal conservation laws* arise from multipliers that do not depend at most on a finite number of derivatives of the dependent variables of a given system.

The approach taken here is to extend the existing similarity methods to systems that are nonlocally related but equivalent to a given system of PDEs. As a starting point, a natural and systematic way of doing this is through the use of admitted local conservation laws. In particular, for any local conservation law (2.26) of a given PDE (2.22), one can construct *an equivalent augmented potential system* P given by

$$\begin{aligned}
&\frac{\partial v}{\partial t} = X(x,t,u,\partial u,\ldots,\partial^r u)\ ,\\
&\frac{\partial v}{\partial x} = -T(x,t,u,\partial u,\ldots,\partial^r u)\ ,\\
&G = (x,t,u,\partial u,\ldots,\partial^k u) = 0\ .
\end{aligned} \tag{3.1}$$

If (u, v) solves the potential system (3.1), then u solves the given PDE (2.22). Conversely, if u solves the given PDE (2.22), then there exists a solution (u, v) of the potential system (3.1) due to the integrability condition $v_{xt} = v_{tx}$ being satisfied from the local conservation law (2.26). But this equivalence relationship between the given PDE (2.22) and the related potential system (3.1) is *nonlocal* and *non-invertible* since for any solution u of (2.22), if (u, v) solves the potential system (3.1), then so does $(u, v + C)$for any constant C.

As a consequence of the equivalence relationship, it follows that a symmetry or conservation law of the given PDE (2.22) yields a corresponding symmetry or conservation of the potential system (3.1); conversely, a symmetry or conservation law of the potential system (3.1) yields a corresponding symmetry or conservation law of the given PDE (2.22).

In particular, suppose a point symmetry

$$X = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \omega(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v} \tag{3.2}$$

is admitted by the potential system (3.1). Then it follows that the point symmetry (3.2) yields a nonlocal symmetry of the given PDE (2.22) if and only if

$$(\xi_v)^2 + (\tau_r)^2 + (\omega_v)^2 \not\equiv 0\ , \tag{3.3}$$

i.e., the infinitesimals of the variables (x, t, u) of the given PDE (2.22) have an essential dependence on the potential variable v.

Hence through a local conservation law of a given PDE (2.22), a nonlocal symmetry of (2.22) can be obtained through a local symmetry admitted by the related potential system (3.1). Conversely, a local symmetry of the given PDE (2.22) could yield a nonlocal symmetry of the potential system (3.1) if there is no corresponding local symmetry of the potential system (2.1).

3.1 Construction of further nonlocally related systems

Now suppose, there are n multipliers $\mu_i(x, t, U, \partial U, \ldots, \partial^q U)\|_{i=1}^n$ yielding n independent local conservation laws of the given PDE (2.22). Let v_i be the potential variable arising from multiplier μ_i. Then we obtain n *singlet* potential systems P^i, $i = 1, \ldots, n$. Moreover, further distinct nonlocally but equivalent potential systems can arise by considering *couplets* $\left\{P^i,\ P^j\right\}_{i,j=1}^n$ with two potential variables; *triplets* $\left\{P^i,\ P^j,\ P^k\right\}_{i,j,k=1}^n$ with three potential variables; … ; and the *n-plet* $\{P^i, \ldots, P^n\}$ with n potential variables. Consequently, from n local conservation laws, we can obtain up to $2^n - 1$ distinct potential systems!

Furthermore, starting from *any* of these $2^n - 1$ potential systems, one can continue the process and if it has N "local" conservation laws, we can obtain up to 2^N-1 further distinct potential systems, etc. Moreover, one can tell in advance in such a further extension if it is possible to obtain further potential systems. In particular, if one has obtained all possible local conservation laws at any stage and has constructed all corresponding multiplet potential systems, then one can show that if the multipliers depend only on the independent variables (x, t) then no new distinct nonlocally related potential system can be obtained [Bluman, Cheviakov and Ivanova (2006)].

Local symmetries or local conservation laws of any constructed potential system could yield new calculable nonlocal symmetries or new calculable nonlocal conservation laws for any of the other potential systems as well as for the given PDE (2.22).

Suppose we are given a system of PDEs $\boldsymbol{S}\{x, t, u_1, \ldots, u_M\} = 0$ with the indicated M dependent variables. A subsystem excluding one of the dependent variables, say u_M, denoted by $\underline{\boldsymbol{S}}\{x, t, u_1, \ldots, u_{M-1}\} = 0$ is *nonlocally related* to the given system $\boldsymbol{S}\{x, t, u, \ldots, u_M\} = 0$ if u_M cannot be directly expressed from the equations of $\boldsymbol{S}\{x, t, u_1, \ldots, u_M\} = 0$ in terms of x, t, the remaining dependent variables $u_1, \ldots, u_{M-1}$, and a finite number of their derivatives. Subsystems for consideration can arise following an interchange of dependent and independent variables of the given system $\boldsymbol{S}\{x, t, u_1, \ldots, u_M\} = 0$.

The above-mentioned ways of constructing nonlocally related systems lead to trees of nonlocally related (but equivalent) systems arising from conservation laws and subsystems. Each system in any such tree is equivalent in the sense that the solution set for any system in a tree can be found from the solution set for any other system in the tree. Moreover, due to the equivalence of the solution sets and the nonlocal relationship, it follows that any coordinate-independent method of analysis (quantitative, analytical, numerical, perturbation, etc.) when applied to any system in the tree may yield simpler computations and/or results that cannot be obtained when the method is directly applied to the given system. Note also that the "given" system could be any system in a tree!!

4 Examples

Now we illustrate the construction of trees of nonlocally related systems and concomitant calculations of nonlocal symmetries and nonlocal conservation laws through three examples involving the nonlinear wave equation, the nonlinear telegraph equation and the planar gas dynamics equations.

4.1 Nonlinear wave equation

Suppose the given PDE is the nonlinear wave equation

$$\boldsymbol{U}\{x, t, u\} \ : \qquad u_{tt} = \left(c^2(u) u_x\right)_x \ . \tag{4.1}$$

Directly, one obtains the singlet potential system (multiplier is 1)

$$\boldsymbol{UV}\{x, t, u, v\} \ : \begin{cases} v_x - u_t = 0 \ , \\ v_t - c^2(u) u_x = 0 \ . \end{cases} \tag{4.2}$$

Through the invertible point transformation (hodograph transformation) $x = x(u, v), t = t(u, v)$, the $\boldsymbol{UV}$ potential system (4.2) becomes the locally related system

$$\boldsymbol{XT}\{u, v, x, t\} \ : \begin{cases} x_v - t_u = 0 \ , \\ x_u - c^2(u) t_v = 0 \ . \end{cases} \tag{4.3}$$

One can show that there are only three more multipliers of the form $\mu(x,t,u) = xt, x, t$ that yield local conservation laws for the given PDE (4.1) for an *arbitrary* wave speed $c(u)$. These three multipliers yield three more singlet potential systems given by

$$\boldsymbol{UA}\{x,t,u,a\} \ : \left\{ \begin{array}{l} a_x - x[tu_t - u] = 0 \ , \\ a_t - t\big[xc^2(u)u_x - \int c^2(u)du\big] = 0 \ , \end{array} \right. \tag{4.4}$$

$$\boldsymbol{UB}\{x,t,u,b\} \ : \left\{ \begin{array}{l} b_x - xu_t = 0 \ , \\ b_t - \big[xc^2(u)u_x - \int c^2(u)du\big] = 0 \ , \end{array} \right. \tag{4.5}$$

and

$$\boldsymbol{UW}\{x,t,u,w\} \ : \left\{ \begin{array}{l} w_x - [tu_t - u] = 0 \ , \\ w_t - t\int c^2(u)du = 0 \ , \end{array} \right. \tag{4.6}$$

with potential variables a, b, and w, respectively.

Obvious nonlocally related subsystems arising from potential system (4.2) through (4.3) include

$$\boldsymbol{T}(u,v,t) : t_{vv} - c^{-2}(u)t_{uu} = 0 \ , \tag{4.7}$$

and

$$\boldsymbol{X}(u,v,t) : x_{vv} - \big(c^{-2}(u)x_u\big)_u = 0 \ . \tag{4.8}$$

The symmetry classifications of PDEs (4.7) and (4.8) are similar modulo equivalence transformations.

Consider PDE (4.7). One can show that there are only four multipliers of the form $\mu(u,v,t) = c^2(u),\ uc^2(u),\ vc^2(u),\ uvc^2(u)$ that yield local conservation laws for PDE (4.7) for an arbitrary wave speed $c(u)$. Note that such conservation laws are nonlocal conservation laws for the given nonlinear wave equation (4.1). The resulting new singlet potential systems include

$$\boldsymbol{TP}\{u,v,t,p\} \ : \left\{ \begin{array}{l} p_v - (ut_u - t) = 0 \ , \\ p_u - uc^2(u)t_v = 0 \ , \end{array} \right. \tag{4.9}$$

$$\boldsymbol{TQ}\{u,v,t,q\} \ : \left\{ \begin{array}{l} q_v - vt_u = 0 \ , \\ q_u + c^2(u)(t - vt_v) = 0 \ , \end{array} \right. \tag{4.10}$$

and

$$\boldsymbol{TR}\{u,v,t,r\} \ : \left\{ \begin{array}{l} r_v - v(ut_u - t) = 0 \ , \\ r_u - uc^2(u)\,[vt_v - t] = 0 \ , \end{array} \right. \tag{4.11}$$

with respective potential variables p, q and r.

The point symmetry classification of the nonlinear wave equation was given in Ames, Lohner and Adams (1981); the point symmetry classifications of potential system (4.3) and subsystem (4.7) were given in Bluman and Kumei (1987); partial point symmetry classifications of the potential systems (4.9) and (4.10) can be adapted from results presented in Ma (1990). All of these results are included in Bluman and Cheviakov (2007) where, in addition, the complete point symmetry classifications of potential systems (4.4)-(4.6) and (4.9)-(4.11) are given. Many nonlocal symmetries for the nonlinear wave equation are found from each of these nonlocally related systems in terms of specific forms of the nonlinear wave speed $c(u)$. In particular, the following new nonlocal symmetries for the nonlinear wave equation (4.1) were found in Bluman and Cheviakov (2007) that had not been found previously:

From potential system (4.5), setting $F(u) = \int c^2(u)du$, one finds that if $F(u)$ satisfies the ODE

$$\frac{F''}{F'^2} = \frac{4F + 2C_1}{(F + C_2)^2 + C_3} \,,$$

with arbitrary constants C_1, C_2, C_3, then the potential system (4.5) admits the point symmetry

$$X = (F + C_1)\, x\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + \frac{(F + C_2)^2 + C_3}{F'}\frac{\partial}{\partial u} + \big(2C_2 b - (C_2^2 + C_3)t\big)\frac{\partial}{\partial b} \qquad (4.12)$$

that is a nonlocal symmetry of the nonlinear wave equation (4.1).

From potential system (4.6), if $c(u)$ satisfies the ODE

$$\frac{c'}{c} = -\frac{2u + C_1}{u^2 + C_2} \,,$$

with arbitrary constants C_1, C_2, then the potential system (4.6) admits the point symmetry

$$X = w\frac{\partial}{\partial x} + (u + C_1)t\frac{\partial}{\partial t} + (u^2 + C_2)\frac{\partial}{\partial u} - C_2 x\frac{\partial}{\partial w} \,,$$

that is a nonlocal symmetry of the nonlinear wave equation (4.1).

From potential system (4.9), for

$$c(u) = u^{-2}e^{1/u} \,,$$

the potential system (4.9) admits the point symmetries

$$X_1 = \big(pu - 2tv(u+1)\big)\frac{\partial}{\partial t} - 2u^2 v\frac{\partial}{\partial u} + (u^2 + e^{2/u})\frac{\partial}{\partial v} + tu^{-1}e^{2/u}\frac{\partial}{\partial p} \,,$$
$$X_2 = t(u+1)\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \,,$$

that are both nonlocal symmetries of the nonlinear wave equation (4.1).

For potential system (4.11), new nonlocal symmetries are found for (4.1) from point symmetries admitted by (4.11) when $c(u) = u^{-4/3}$. Consequently, one obtains a (far from exhaustive) tree of nonlocally related systems for the nonlinear wave equation $\boldsymbol{U}$ (4.1) for an arbitrary wave speed $c(u)$ exhibited in Figure 1.

4.2 Nonlinear telegraph equation

As a second example, consider the nonlinear telegraph (NLT) equation given by

$$\boldsymbol{U}\{x,t,u\}: \quad u_{tt} - \big(F(u)u_x\big)_x - \big(G(u)\big)_x = 0 \,. \qquad (4.13)$$

The local conservation laws of the NLT PDE (4.13) resulting from multipliers of the form $\mu(x,t,U)$ were classified in Bluman, Cheviakov and Ivanova (2006) and the resulting singlet potential systems can be summarized as follows.

Case (a) $F(u)$, $G(u)$ both arbitrary. Here there are two singlet potential systems

$$\boldsymbol{UV}_1\{x,t,u,v_1\}\ :\begin{cases} v_{1x}-u_t=0\ , \\ v_{1t}-\big(F(u)u_x+G(u)\big)=0\ ; \end{cases} \tag{4.14}$$

$$\boldsymbol{UV}_2\{x,t,u,v_2\}\ :\begin{cases} v_{2x}-(tu_t-u)=0\ , \\ v_{2t}-t\big(F(u)u_x+G(u)\big)=0\ . \end{cases} \tag{4.15}$$

Case (b) $G(u)$ arbitrary, $F(u)=G'(u)$. In addition to (4.14) and (4.15), there are two more singlet potential systems

$$\boldsymbol{UB}_3\{x,t,u,b_3\}\ :\begin{cases} b_{3x}-e^x u_t=0\ , \\ b_{3t}-e^x F(u)u_x=0\ ; \end{cases} \tag{4.16}$$

$$\boldsymbol{UB}_4\{x,t,u,b_4\}\ :\begin{cases} b_{4x}-e^x(tu_t-u)=0\ , \\ b_{4t}-te^x F(u)u_x=0\ . \end{cases} \tag{4.17}$$

Case (c) $F(u)$ arbitrary, $G(u)=u$. In addition to (4.14) and (4.15), there are two more singlet potential systems

$$\boldsymbol{UC}_3\{x,t,u,c_3\}\ :\begin{cases} c_{3x}-\left(\left(x-\frac{1}{2}\,t^2\right)u_t+tu\right)=0\ , \\ c_{3t}-\left(x-\frac{1}{2}\,t^2\right)\left(F(u)u_x+u\right)+\int F(u)du=0\ ; \end{cases} \tag{4.18}$$

$$\boldsymbol{UC}_4\{x,t,u,c_4\}\ :\begin{cases} c_{4x}+\left(\frac{1}{6}\,t^3-tx\right)u_t+\left(x-\frac{1}{2}\,t^2\right)u=0\ , \\ c_{4t}+\left(\frac{1}{6}\,t^3-tx\right)\left(F(u)u_x+u\right)+t\int F(u)du=0\ . \end{cases} \tag{4.19}$$

A resulting tree of nonlocally related potential systems for Case (b) is exhibited in Figure 2.

The complete conservation law classification of potential system (4.14) is given in Bluman and Temuerchaolu (2005) for multipliers depending on (x,t,U,V_1). The point symmetry classification of potential system (4.14) is given in Bluman, Temuerchaolu and Sahadevan (2005). Many of these symmetries yield nonlocal symmetries for the NLT equation (4.13).

4.3 Planar gas dynamics equations

As a final example, consider the planar gas dynamics (PGD) equations. In the *Eulerian description*, one has the *Euler system* given by

$$\boldsymbol{E}\{x,t,v,p,\rho\}\ :\begin{cases} \rho_t+(\rho v)_x=0\ , \\ \rho(v_t+vv_x)+p_x=0\ , \\ \rho(p_t+vp_x)+B(p,\rho^{-1})v_x=0\ , \end{cases} \tag{4.20}$$

where in terms of the entropy density function $S(p,\rho)$, the constitutive function $B(p,\rho^{-1})$ is given by

$$B(p,\rho^{-1}) = -\rho^2\,\frac{S_\rho}{S_p}\;.$$

In the *Lagrangian description*, in terms of Lagrange mass coordinates $s=t$, $y=\int_{x_0}^{x}\rho(\xi)d\xi$, one has the *Lagrange system* given by

$$\boldsymbol{L}\{y,s,v,p,q\}\;:\begin{cases} q_s - v_y = 0\;,\\ v_s + p_y = 0\;,\\ p_s + B(p,q)v_y = 0\;,\end{cases} \tag{4.21}$$

where $q=\rho^{-1}$.

We now show that within the potential system framework outlined in Section 3 there is a direct connection between the Euler and Lagrange systems. As well, we derive other equivalent descriptions!

We use the Euler system (4.20) as the given system. The first equation of (4.20) is a conservation law as written and through it, we introduce a potential variable r and obtain the potential system

$$\boldsymbol{G}\{x,t,v,p,\rho,r\}\;:\begin{cases} r_x - \rho = 0\;,\\ r_t + \rho v = 0\;,\\ \rho(v_t + v v_x) + p_x = 0\;,\\ \rho(p_t + v p_x) + B(p,\rho^{-1})v_x = 0\;.\end{cases} \tag{4.22}$$

In order to obtain a nonlocally related subsystem of (4.22), we first consider an interchange of dependent and independent variables in (4.22) with $r=y$ and $t=s$ as independent variables; x,v,p,ρ as dependent variables and let $q=1/\rho$ to obtain the invertibly equivalent system

$$\boldsymbol{G}\{y,s,x,v,p,\rho\}\;:\begin{cases} x_y - q = 0\;,\\ x_s - v = 0\;,\\ v_s + p_y = 0\;,\\ p_s + B(p,q)v_y = 0\;.\end{cases} \tag{4.23}$$

A nonlocally related subsystem of (4.23) is obtained by excluding x through $x_{ys}=x_{sy}$ to obtain the *Lagrange system* (4.21). A second conservation law of the Euler system (4.20) obtained with the multipliers

$$(\mu_1,\mu_2,\mu_3) = (v,1,0)$$

yields a second potential variable w. The couplet potential system containing potential variables r and w is given by

$$\boldsymbol{W}\{x,t,v,p,\rho,r,w\}\;:\begin{cases} r_x - \rho = 0\;,\\ r_t + \rho v = 0\;,\\ w_x + r_t = 0\;,\\ w_t + p + v w_x = 0\;,\\ \rho(p_t + v p_x) + B(p,\rho^{-1})v_x = 0\;.\end{cases} \tag{4.24}$$

From the third PDE of (4.24), one can introduce a third potential variable z and obtain a potential system

$$\boldsymbol{Z}\{x,t,v,p,\rho,r,w,z\} \ : \begin{cases} r_x - \rho = 0 \ , \\ r_t + \rho v = 0 \ , \\ z_t - w = 0 \ , \\ z_x + r = 0 \ , \\ w_t + p + v w_x = 0 \ , \\ \rho(p_t + v p_x) + B(p, \rho^{-1}) v_x = 0 \ . \end{cases} \tag{4.25}$$

The Lagrange system (4.21) has a nonlocally related subsystem

$$\underline{\boldsymbol{L}}\{y,s,p,q\} \ : \begin{cases} q_{ss} + p_{yy} = 0 \ , \\ p_s + B(p,q) q_s = 0 \ . \end{cases} \tag{4.26}$$

Starting from the Lagrange system (4.21), one can obtain three singlet potential systems from the three sets of multipliers

$$\big(\mu_1(y,s), \mu_2(y,s), \mu_3(y,s)\big) = (1,0,0), (0,1,0), (y,s,0) \ .$$

In particular, one obtains potential systems (the first one is potential system (4.23))

$$\boldsymbol{G} \Leftrightarrow \boldsymbol{G}_0 = \boldsymbol{LW}_1\{y,s,v,p,q,w_1\} \ : \begin{cases} w_{1y} - q = 0 \ , \\ w_{1s} - v = 0 \ , \\ v_s + p_y = 0 \ , \\ p_s + B(p,q) v_y = 0 \ ; \end{cases}$$

$$\boldsymbol{LW}_2\{y,s,v,p,q,w_2\} \ : \begin{cases} q_s - v_y = 0 \ , \\ w_{2y} - v = 0 \ , \\ w_{2s} + p = 0 \ , \\ p_s + B(p,q) v_y = 0 \ ; \end{cases} \tag{4.27}$$

$$\boldsymbol{LW}_3\{y,s,v,p,q,w_3\} \ : \begin{cases} w_{3y} - sv - yq = 0 \ , \\ w_{3s} + sp - yv = 0 \ , \\ v_s + p_y = 0 \ , \\ p_s + B(p,q) v_y = 0 \ . \end{cases} \tag{4.28}$$

Corresponding trees of nonlocally related systems for the PGD equations are exhibited in Figures 3 and 4. Note that the tree for the Lagrange system (4.21) exhibited in Figure 4 can be added to that of Figure 3 to yield a further extended tree.

Two more conservation laws arise for the Lagrange system (4.21) for multipliers of the form

$$\mu_i(y,s,V,Q,P), \ i = 1,2,3.$$

In particular,

$$\mu_1 = ay - \beta P + B(P,Q)\mu_3 + \delta \ ,$$
$$\mu_2 = as + \beta V + v \ ,$$
$$\mu_3 = \mu_3(y,P,Q) \ ,$$

where a, β, v, δ are arbitrary constants and $\mu_3(y, P, Q)$ is any solution of PDE

$$\frac{\partial \mu_3}{\partial Q} - \frac{\partial}{\partial P}\left(B(P,Q)\mu_3\right) + \beta = 0 \,. \tag{4.29}$$

These two extra conservation laws arise (for *arbitrary* constitutive function $B(p, q)$) from the conservation of energy equation

$$\frac{\partial}{\partial s}\left(\frac{1}{2}\,v^2 + K(p,q)\right) + \frac{\partial}{\partial y}\,(pv) = 0 \,,$$

where $K(p, q)$ satisfies

$$K_q - B(p,q)K_p + p = 0 \,,$$

and from the conservation of entropy equation

$$\frac{\partial}{\partial s}\,S(p,q) = 0 \,,$$

where $S(p, q)$ satisfies

$$S_q - B(p,q)S_p = 0 \,.$$

These two extra conservation laws yield two more singlet potential systems and hence further extend the trees of equivalent nonlocally related systems exhibited in Figures 3 and 4.

In Bluman and Cheviakov (2005), new nonlocal symmetries for both the Euler system (4.20) and Lagrange system (4.21) were constructed for a Chaplygin gas for which $B(p, q) = -p/q$. In Bluman, Cheviakov and Ivanova (2006), invariant solutions arising from a nonlocal symmetry were constructed for the Lagrange system (4.21) in the case of a polytropic gas where $B(p, q) = -(p \ln p)/q$.

Heuristic approaches have been used to obtain nonlocal symmetries for gas dynamics equations [Akhatov, Gazizov and Ibragimov (1991), Sjöberg and Mahomed (2004)]. One can show that all nonlocally related systems considered in these two papers arise within the framework presented in Section 3.

Trees of potential systems have been found for a class of diffusion-convection equations [Popovych and Ivanova (2005)].

5 Concluding remarks

In this paper, an attempt has been made to summarise some recent work on extending similarity methods through use of nonlocally related systems. For the three considered examples as well as many other well-known systems of PDEs, one should be able to construct further extended trees of nonlocally related systems through the procedures presented in Section 3. Consequently, through calculation of point symmetries of any system in such trees, one may be able to find further nonlocal symmetries for PDE systems of interest.

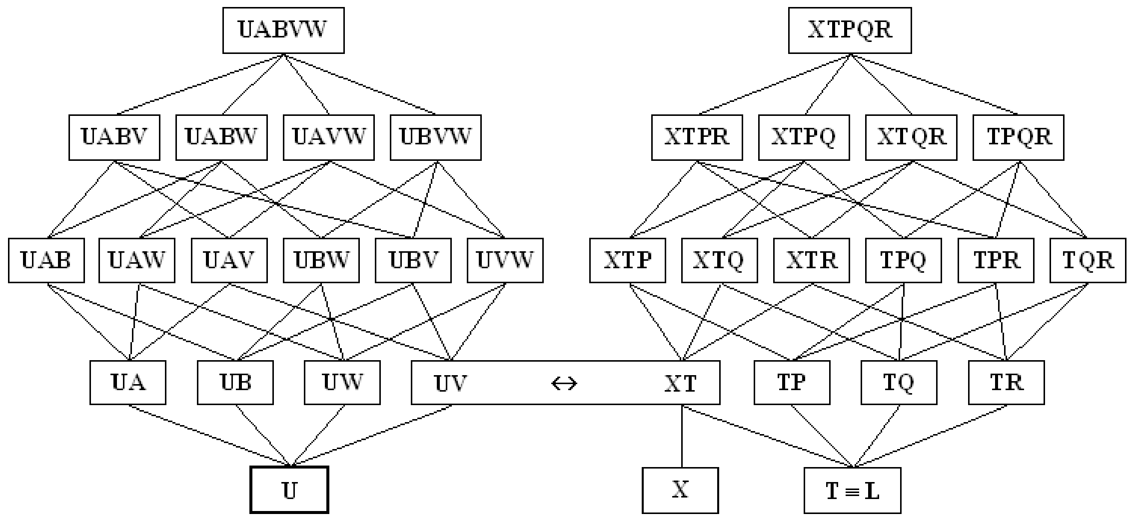

Figure 1: A tree of nonlocally related systems for the nonlinear wave equation (4.1) for an arbitrary wave speed $c(u)$.

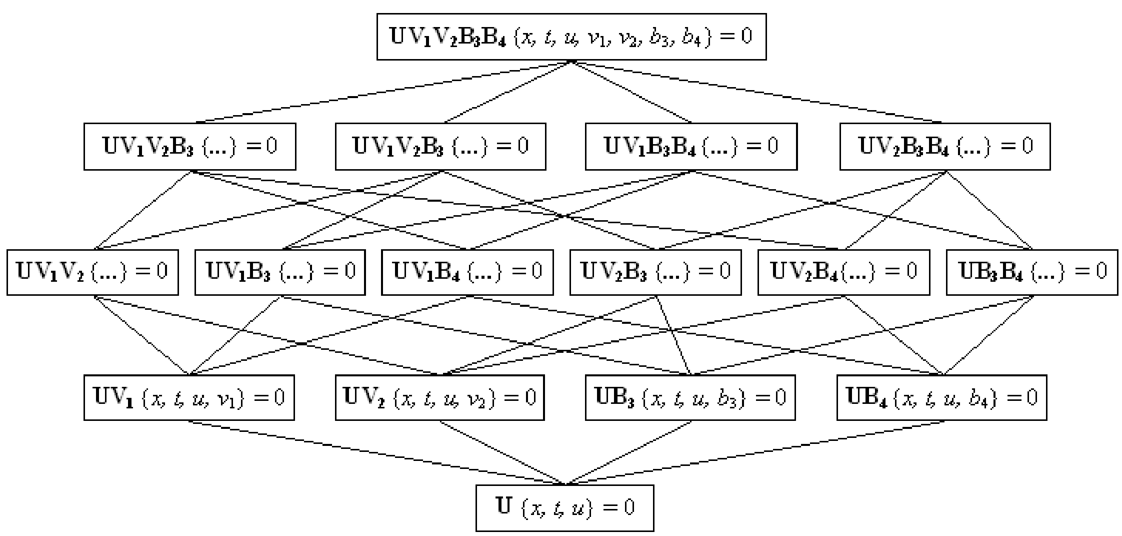

Figure 2: A tree of nonlocally related systems for the nonlinear telegraph equation (4.13) for arbitrary $G(u)$, $F(u) = G'(u)$.

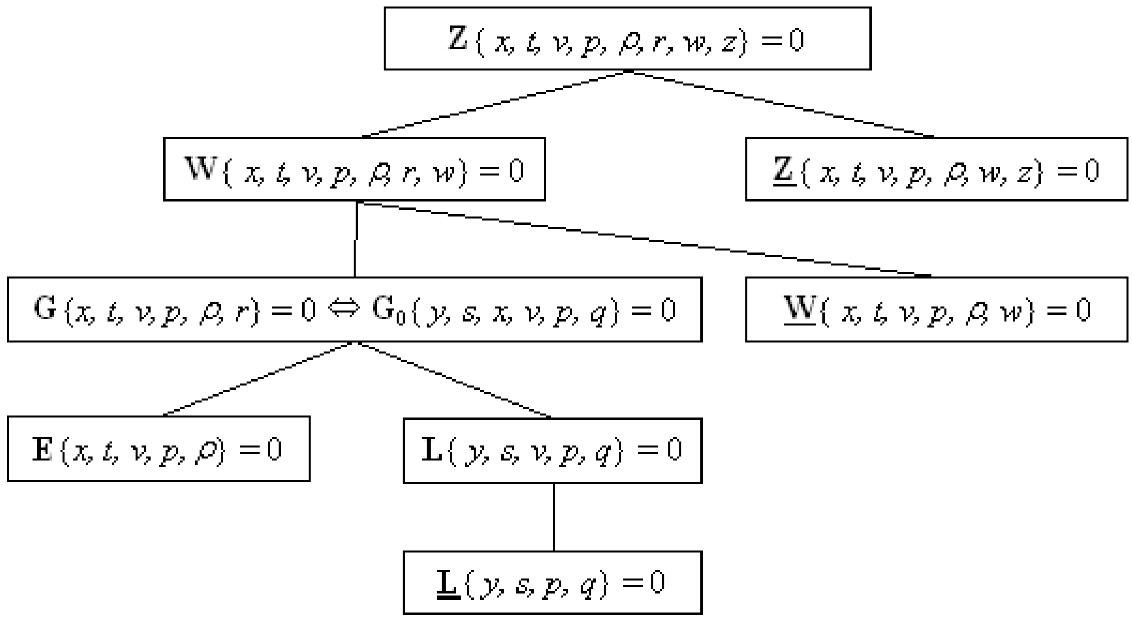

Figure 3: A tree of nonlocally related systems, which includes the Euler and Lagrange systems, for the PGD equations for arbitrary $B(p,q)\,;\ q = 1/\rho$.

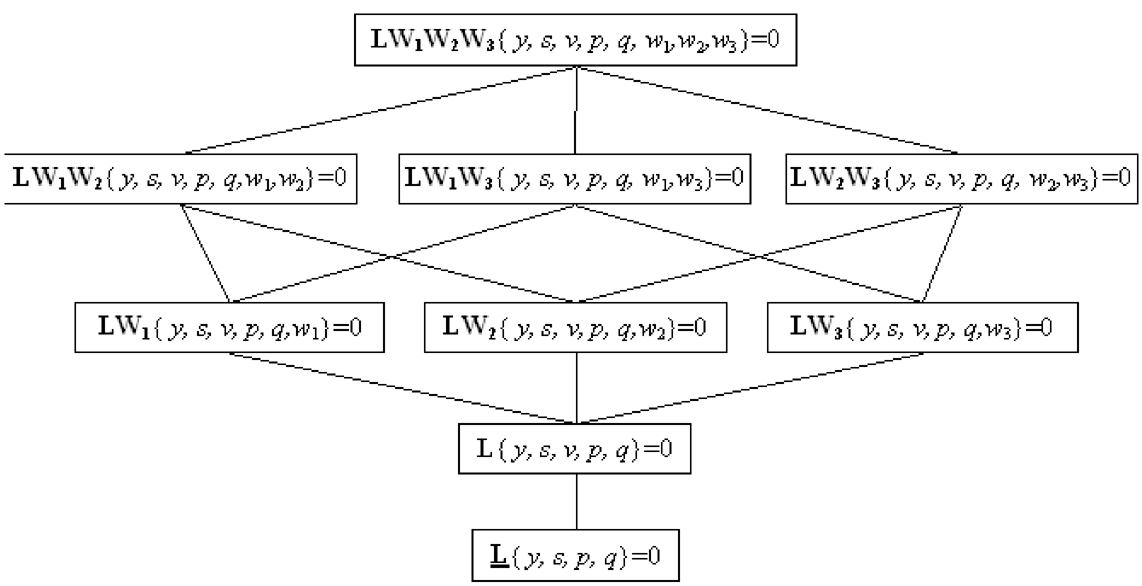

Figure 4: A tree of nonlocally related systems for the Lagrange PGD system for arbitrary $B(p,q)$.

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