Exact solutions for wave equations of two-layered media with smooth transition

George Bluman
Department of Mathematics and Institute of Applied Mathematics, University of British Columbia, Vancouver, Canada V6T 1Y4

Sukeyuki Kumei
The Faculty of Textile Science and Technology, Shinshu University, 3-15-1 Tokida, Ueda, Nagano ken, Japan

(Received 5 January 1987; accepted for publication 13 August 1987)

The wave equation $c^2(x)u_{xx} - u_{tt} = 0$ is solved for wave speeds $c(x)$ corresponding to two-layered media with smooth transition from layer to layer. The wave speed $c(x)$ has four free parameters to fit a given medium. Solutions are constructed from invariant solutions of a related system of first-order partial differential equations that admit a four-parameter symmetry group. These solutions are superposed to solve general initial value problems for data with compact support; the computation of the superposition coefficients uses elementary Fourier analysis. Solutions are illustrated for various initial conditions.

I. INTRODUCTION

In a previous paper[1] we classified all wave equations of the form

$$c^2(x)u_{xx} - u_{tt} = 0, \tag{1.1}$$

which are solvable by group theoretical methods. In particular, we showed that the system of partial differential equations

$$v_t = u_x, \quad u_t = c^2(x)v_x \tag{1.2}$$

equivalent to Eq. (1.1), admits a maximal four-parameter Lie group of point transformations if and only if the wave speed $c(x)$ satisfies the ordinary differential equation

$$cc'(c/c')'' = \text{const} = \mu. \tag{1.3}$$

If $\mu = 0$, the solution of Eq. (1.3) reduces to either $c(x) = e^x$ or $c(x) = x^A$, where A is an arbitrary constant. In Ref. 1 we constructed the corresponding invariant solutions of (1.2).

If $\mu \neq 0$, Eq. (1.3) reduces to one of the following four standard forms[1]:

$\boxed{\mu = 1}$

$$c' = \nu^{-1}\sin(\nu\log c); \tag{1.4}$$

$$c' = \nu^{-1}\sinh(\nu\log c); \tag{1.5}$$

$$c' = \log c; \tag{1.6}$$

$\boxed{\mu = -1}$

$$c' = \nu^{-1}\cosh(\nu\log c); \tag{1.7}$$

where $\nu \neq 0$ is an arbitrary constant. Solutions of (1.1) and (1.2) are discussed in Ref. 2 for $c(x)$ satisfying (1.5) or (1.7) with $\nu = \frac{1}{2}$.

If $c(x) = \phi(x,\nu)$ is a solution of any one of the equations (1.4)–(1.7) then the corresponding general solution of Eq. (1.3) is given by

$$c(x) = K\phi(Lx + M,\nu), \tag{1.8}$$

where $K^2L^2 = |\mu|$ for any constants $\{L,M,\nu\}$.

For each of the equations (1.4)–(1.7) solutions $c(x)$ are monotone functions of x; $c(x)$ is bounded on $(-\infty,\infty)$ if and only if $c(x)$ satisfies Eq. (1.4). Such a bounded $c(x)$ has a smooth simple jump (cf. Fig. 1). This corresponds to wave propagation in a two-layered stratified medium with a smooth transition from layer to layer.

In the rest of this paper we construct various invariant solutions of system (1.2) and hence solutions of (1.1), where the wave speed $c(x)$ satisfies (1.4); without loss of generality $\nu > 0$. We show how to solve general initial value problems by a superposition of these invariant solutions. We illustrate our results by solving initial value problems for initial humps of varying shape and location.

II. PROPERTIES OF $c(x)$

Say $c(x)$ solves (1.4). Then $|c'(x)| \leqslant 1/\nu$, and $c'(x) = 0$ if and only if

$$c(x) = e^{k\pi/\nu}, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{2.1}$$

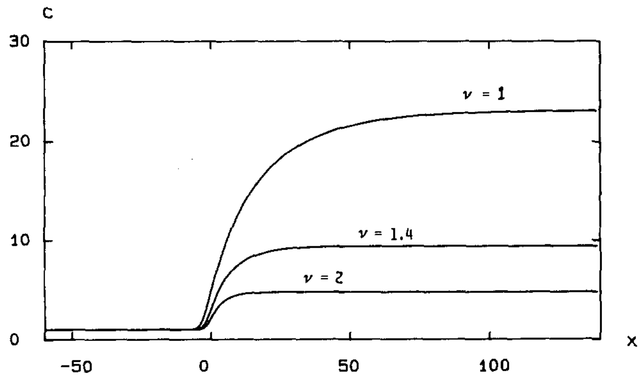

FIG. 1. Profile of $c(x) = \Phi(x,\nu)$.

 0022-2488/88/010086-11$02.50

Now consider the region where

$$1 < c(x) < e^{\pi/\nu}. \tag{2.2}$$

In this strip $c'(x) > 0$ and the inflection point $x = x^*$ occurs where $c(x^*) = e^{\pi/2\nu}$; $c'(x^*) = 1/\nu$. Equation (1.4) leads to

$$\lim_{\epsilon\to 0^+} \int_{e^{\pi/2\nu}}^{e^{\pi/\nu}-\epsilon} \frac{dc}{\sin(\nu \log c)} = +\infty, \tag{2.3}$$

and

$$\lim_{\epsilon\to 0^+} \int_{1+\epsilon}^{e^{\pi/2\nu}} \frac{dc}{\sin(\nu \log c)} = +\infty. \tag{2.4}$$

Hence it follows that $\lim_{x\to+\infty} c(x) = e^{\pi/\nu}$, $\lim_{x\to-\infty} c(x) = 1$. Thus $c = 1$, $c = e^{\pi/\nu}$ are horizontal asymptotes for $c(x)$ in the strip (2.2). Since Eq. (1.4) is invariant under translation in x, without loss of generality we can set $x^* = 0$.

Now let

$$c(x) = \Phi(x,\nu) \tag{2.5}$$

be the solution of Eq. (1.4) with properties

$$\lim_{x\to-\infty} \Phi(x,\nu) = 1, \tag{2.6}$$

$$\lim_{x\to+\infty} \Phi(x,\nu) = e^{\pi/\nu}, \tag{2.7}$$

$$\Phi(0,\nu) = e^{\pi/2\nu}. \tag{2.8}$$

One can show that Eq. (1.4) has solutions

$$c(x) = e^{-n\pi/\nu}\Phi((-1)^n e^{n\pi/\nu}x,\nu) \tag{2.9}$$

on the horizontal strip

$$e^{n\pi/\nu} < c(x) < e^{(n+1)\pi/\nu}, \tag{2.10}$$

$n = 0, \pm 1, \pm 2, \ldots$.

From property (1.8) it follows that each strip solution leads to the same general solution of Eq. (1.3). Thus from now on we will only consider the solution $c(x) = \Phi(x,\nu)$ of Eq. (1.4).

Graphs of $\Phi(x,\nu)$ and $(d/dx)\Phi(x,\nu)$ are given in Figs. 1 and 2, respectively, for $\nu = 1, 1.4, 2$.

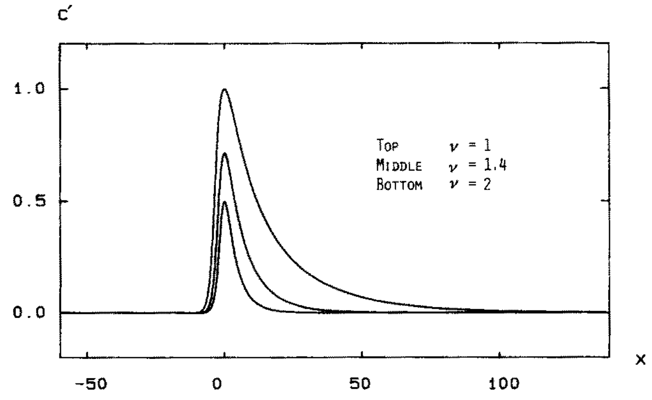

FIG. 2. $c' = \Phi'(x,\nu)$.

Here $\Phi(x,\nu)$ has asymptotic properties,

$$\Phi(x,\nu) = 1 + C^-(\nu)e^x + o(e^x) \quad \text{as } x\to-\infty, \tag{2.11}$$

$$\Phi(x,\nu) = e^{\pi/\nu}[1 - C^+(\nu)e^{-(e^{-\pi/\nu})x}] + o(e^{-(e^{-\pi/\nu})x}) \quad \text{as } x\to+\infty, \tag{2.12}$$

$$\Phi(x,\nu) = e^{\pi/2\nu} + x/\nu + o(x^2) \quad \text{as } x\to 0, \tag{2.13}$$

for some positive constants $\{C^-(\nu), C^+(\nu)\}$.

To obtain a bounded monotonically increasing solution $c(x)$ of Eq. (1.3) with the properties

$$\lim_{x\to-\infty} c(x) = c_1, \tag{2.14}$$

$$\lim_{x\to+\infty} c(x) = c_2, \tag{2.15}$$

and

$$\max_{x\in(-\infty,\infty)} c'(x) = m, \tag{2.16}$$

where $\{c_1, c_2, m\}$ are arbitrary positive constants with $0 < c_1 < c_2$, we set in (1.8),

$$K = c_1, \quad L = (m/c_1)\nu^*, \quad \nu = \nu^* = \pi(\log c_2/c_1)^{-1}. \tag{2.17}$$

The general solution of Eq. (1.3) satisfying (2.14)–(2.16) is

$$c(x) = c_1\Phi((m/c_1)\nu^* x + M, \nu^*), \tag{2.18}$$

where M is an arbitrary constant.

The width of the transition region in x is $O((c_2 - c_1)/m)$. Since $\Phi(x,\nu)$ exponentially approaches its horizontal asymptotes, a wave speed $c(x)$, represented by (2.18), effectively approximates a two-layered medium. The transition between layers can be as abrupt as one wishes.

III. INVARIANCE PROPERTIES OF SYSTEM (1.2)

As shown in Ref. 1, when $c(x)$ satisfies (1.3) for $\mu > 0$, the system (1.2) admits the four-parameter $\{p,q,r,s\}$ Lie group of point transformations

$$\begin{aligned} X &= x + \epsilon\xi(x,t) + O(\epsilon^2),\\ T &= t + \epsilon\tau(x,t) + O(\epsilon^2),\\ U &= u + \epsilon[i(x,t)u + j(x,t)v] + O(\epsilon^2),\\ V &= v + \epsilon[k(x,t)v + l(x,t)u] + O(\epsilon^2), \end{aligned} \tag{3.1}$$

where in terms of

$$\beta(t) = pe^t - qe^{-t}, \tag{3.2}$$

$\{\xi,\tau,i,j,k,l\}$ are given by

$$\begin{aligned} \xi &= 2\beta'(t)[c(x)/c'(x)],\\ \tau &= 2\beta(t)[(c(x)/c'(x))' - 1] + r,\\ i &= \beta'(t)[2 - (c(x)/c'(x))'] + s,\\ j &= -\beta(t)[c(x)/c'(x)],\\ k &= -\beta'(t)[c(x)/c'(x)]' + s,\\ l &= -\beta(t)[1/c(x)c'(x)]. \end{aligned} \tag{3.3}$$

The group generators for the parameters $\{p,q,r,s\}$, respectively, are

$$L_p = e^t\left[\frac{2c}{c'}\frac{\partial}{\partial x} + 2\left[\left(\frac{c}{c'}\right)' - 1\right]\frac{\partial}{\partial t} + \left[\left[2-\left(\frac{c}{c'}\right)'\right]u - \frac{c}{c'}v\right]\frac{\partial}{\partial u} - \left[\left(\frac{c}{c'}\right)' v + \frac{1}{cc'}u\right]\frac{\partial}{\partial v}\right],$$

$$L_q = e^{-t}\left[\frac{2c}{c'}\frac{\partial}{\partial x} + 2\left[1-\left(\frac{c}{c'}\right)'\right]\frac{\partial}{\partial t} + \left[\left[2-\left(\frac{c}{c'}\right)'\right]u + \frac{c}{c'}v\right]\frac{\partial}{\partial u} - \left[\left(\frac{c}{c'}\right)' v - \frac{1}{cc'}u\right]\frac{\partial}{\partial v}\right], \quad (3.4)$$

$$L_r = \frac{\partial}{\partial t}, \quad L_s = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

The commutators of the Lie algebra are

$$[L_p,L_q] = -8\nu^2 L_r, \quad [L_r,L_p] = L_p,$$
$$[L_r,L_q] = -L_q, \quad [L_p,L_s] = [L_q,L_s] = [L_r,L_s] = 0. \quad (3.5)$$

The global transformation generated by (3.1)–(3.3) is found by solving the characteristic equations

$$\frac{dX}{\xi(X,T)} = \frac{dT}{\tau(X,T)} = \frac{dU}{i(X,T)U + j(X,T)V} = \frac{dV}{k(X,T)V + l(X,T)U} = d\epsilon, \quad (3.6)$$

where

$$X = x, \quad T = t, \quad U = u, \quad V = v, \quad \text{at } \epsilon = 0. \quad (3.7)$$

The global transformation for $r \neq 0$ is obtained from the global transformation for $r = 0$ by letting $t \to t + r$. Without loss of generality we set $r = 0$.

Now let

$$Y = \nu \log c(X), \quad \gamma = 4pq = (\beta')^2 - \beta^2. \quad (3.8)$$

Then the resulting implicit global transformation is

$$z = \beta(T)\sin Y,$$
$$[[c(X)]^{-1/2}U + [c(X)]^{1/2}V]^2 = [Ae^{2\epsilon s}\sin Y][\beta(T)\cos Y + \beta'(T)], \quad (3.9)$$
$$[[c(X)]^{-1/2}U - [c(X)]^{1/2}V]^2 = [Be^{2\epsilon s}\sin Y][\beta(T)\cos Y - \beta'(T)];$$

and

$$(1/\nu\sqrt{-\gamma})\log|\cos Y + (1/\sqrt{-\gamma})\beta'(T)\sin Y| = E - 2\epsilon \quad \text{for } \gamma < 0, \quad (3.10)$$

$$(1/\nu\sqrt{\gamma})\arctan\left[\frac{\sqrt{\gamma}}{\beta'(T)}\cot Y\right] = E - 2\epsilon \quad \text{for } \gamma > 0. \quad (3.11)$$

The integration constants $\{z,A,B,E\}$ are expressed in terms of $\{x,t,u,v\}$ by using the initial condition (3.7). Without loss of generality $\gamma = 1$ if $\gamma > 0$, $\gamma = -1$ if $\gamma < 0$.

Now we construct invariant solutions of system (1.2) for $r = 0$. Let

$$y = \nu \log c(x). \quad (3.12)$$

We choose the invariant

$$z = \beta(t)\sin y \quad (3.13)$$

as our similarity variable.

By setting $A = A(z)$, $B = B(z)$, we obtain from (3.9)–(3.11) invariant solutions of the form

$$u = e^{-s\epsilon(x,t)}[c(x)|\sin y|]^{1/2}[|\beta(t)\cos y + \beta'(t)|^{1/2}A(z) + |\beta(t)\cos y - \beta'(t)|^{1/2}B(z)], \quad (3.14)$$

$$v = e^{-s\epsilon(x,t)}[[c(x)]^{-1}|\sin y|]^{1/2}[|\beta(t)\cos y + \beta'(t)|^{1/2}A(z) - |\beta(t)\cos y - \beta'(t)|^{1/2}B(z)], \quad (3.15)$$

where

$$\epsilon(x,t) = (1/2\nu)\log|\cos y + \beta'(t)\sin y| \quad \text{for } \gamma = -1, \quad (3.16)$$

and

$$\epsilon(x,t) = (1/2\nu)\arctan[\cot y/\beta'(t)] \quad \text{for } \gamma = 1. \quad (3.17)$$

The substitution of (3.14) and (3.15) into the system (1.2) leads to a coupled system of first-order linear ordinary differential equations for $A(z)$ and $B(z)$. The form of these ODE's depends on the signs of γ and $\beta(t)\cos y + \beta'(t)$.

If $\gamma = 1$, then either

$$\beta(t)\cos y + \beta'(t) > 0 \quad \text{and} \quad \beta(t)\cos y - \beta'(t) < 0$$

or

$$\beta(t)\cos y + \beta'(t) < 0 \quad \text{and} \quad \beta(t)\cos y - \beta'(t) > 0$$

for all x, t.

If $\gamma = -1$, then for any given t, both $\beta(t)\cos y + \beta'(t)$ and $\beta(t)\cos y - \beta'(t)$ change sign once as x varies from $-\infty$ to $+\infty$.

It is convenient to let

$$A(z) = (\text{sgn}[\beta(t)\cos y + \beta'(t)])f(z), \quad B(z) = g(z). \quad (3.18)$$

Then $\{f(z),g(z)\}$ satisfy the system

$$2(z^2-1)\frac{df}{dz} + \left[-\frac{s}{\nu} + \left(2 - \frac{s}{\nu}\right)z\right]f - (1/\nu)|z^2-1|^{1/2}g = 0,$$
$$2(z^2-1)\frac{dg}{dz} + \left[\frac{s}{\nu} + \left(2 - \frac{s}{\nu}\right)z\right]g + \frac{1}{\nu}\frac{z^2-1}{|z^2-1|^{1/2}}f = 0, \quad (3.19)$$

if $\gamma = -1$, and satisfy the system

$$2(z^2+1)\frac{df}{dz}+\left[2z-\frac{s}{\nu}\right]f-\frac{1}{\nu}\sqrt{z^2+1}\,g=0,$$
$$2(z^2+1)\frac{dg}{dz}+\left[2z+\frac{s}{\nu}\right]g+\frac{1}{\nu}\sqrt{z^2+1}\,f=0, \qquad (3.20)$$

if $\gamma = 1$.

The invariant solutions for $\gamma = -1$ are not valid for all $t>0$ since (3.19) has a singular point at $z=1$. For the rest of this paper we consider solutions of system (1.2) for $\gamma = 1$.

IV. INVARIANT SOLUTIONS OF SYSTEM (1.2) FOR $\gamma=1$

A. The general solution of (3.20)

Let

$$R = 1/2\nu, \quad \sigma = -s/2\nu. \qquad (4.1)$$

Then $f(z)$ satisfies the equation

$$\frac{d^2f}{dz^2}+\frac{3z}{z^2+1}\frac{df}{dz}+\frac{1}{z^2+1}\left[1+R^2-\frac{\sigma^2+\sigma z}{z^2+1}\right]f=0 \qquad (4.2)$$

and

$$g(z)=\frac{\sqrt{z^2+1}}{R}\left[\frac{df}{dz}+\frac{z+\sigma}{z^2+1}f\right]. \qquad (4.3)$$

The general solution of (4.2) is

$$f(z)=C_1\left(\frac{1-iz}{1+iz}\right)^{i\sigma/2} \times F\left(1+iR,1-iR;\tfrac{3}{2}-i\sigma;\tfrac{1}{2}(1+iz)\right) + C_2(1+iz)^{-1/2}(z^2+1)^{i\sigma/2} \times F\left(\tfrac{1}{2}+i(\sigma+R),\tfrac{1}{2}+i(\sigma-R);\tfrac{1}{2}+i\sigma;\tfrac{1}{2}(1+iz)\right), \qquad (4.4)$$

where $F(a,b;c;z)$ is the hypergeometric function,[3] C_1 and C_2 are arbitrary constants.

Let

$$\Psi(z)=\log(z+\sqrt{z^2+1}) \qquad (4.5)$$

and

$$\tilde{f}(\Psi)=\sqrt{z^2+1}\,f(z).$$

Then (4.2) transforms to

$$\frac{d^2\tilde{f}}{d\Psi^2}+\left[R^2-\frac{\sigma^2+\sigma\sinh\Psi}{\cosh^2\Psi}\right]\tilde{f}=0. \qquad (4.6)$$

B. Closed form solutions of (3.20)

Now we construct closed form solutions of (4.2) and (4.3) for various values of σ. From (4.5) and (4.6) we see that for $\sigma=0$,

$$f=f_0(z;\zeta)=(1/\sqrt{z^2+1})\cos[R\Psi(z)+\zeta] \qquad (4.7)$$

solves (4.2) for any real constant ζ. Correspondingly, from (4.3) one gets

$$g=g_0(z;\zeta)=-(1/\sqrt{z^2+1})\sin[R\Psi(z)+\zeta]. \qquad (4.8)$$

Now consider the raising and lowering operators

$$L^+(\lambda)=\sqrt{z^2+1}\frac{d}{dz}+\frac{(1-2\lambda)z+i}{2\sqrt{z^2+1}}, \qquad (4.9)$$

$$L^-(\lambda)=\sqrt{z^2+1}\frac{d}{dz}+\frac{(1+2\lambda)z-i}{2\sqrt{z^2+1}}. \qquad (4.10)$$

One can show that if $f=f_\lambda(z;\zeta)$ solves (4.2) for $\sigma=-i\lambda$, then

$$f=L^+(\lambda)f_\lambda(z;\zeta) \qquad (4.11)$$

solves (4.2) for $\sigma=-i(\lambda+1)$, and

$$f=L^-(\lambda)f_\lambda(z;\zeta) \qquad (4.12)$$

solves (4.2) for $\sigma=-i(\lambda-1)$.

For $\sigma=-in$, $n=1,2,\ldots$, recursively we can obtain closed form solutions

$$f=f_n(z;\zeta)=L^+(n-1)f_{n-1}(z;\zeta), \quad n=1,2,\ldots, \qquad (4.13)$$

for (4.2) from $f_0(z;\zeta)$ defined by (4.7).

From (4.3), the corresponding solution is

$$g=g_n(z;\zeta)=\frac{\sqrt{z^2+1}}{R}\left[\frac{df_n(z;\zeta)}{dz}+\frac{z-in}{z^2+1}f_n(z;\zeta)\right]. \qquad (4.14)$$

From (4.7)–(4.11), it then follows that

$$L^-(n)f_n(z;\zeta)=-\tfrac{1}{4}[(2n-1)^2+4R^2]f_{n-1}(z;\zeta), \quad n=1,2,\ldots. \qquad (4.15)$$

Using (4.13)–(4.15), one can show that

$$\begin{bmatrix}f_{n+1}(z;\zeta)\\ g_{n+1}(z;\zeta)\end{bmatrix}=\begin{pmatrix}a(n,z) & R\\ -R & \overline{a(n,z)}\end{pmatrix}\begin{bmatrix}f_n(z;\zeta)\\ g_n(z;\zeta)\end{bmatrix}, \qquad (4.16)$$

where

$$a(n,z)=-(n+\tfrac{1}{2})[(z-i)/\sqrt{z^2+1}], \quad n=0,1,2,\ldots,$$

and $\overline{a(n,z)}$ is the complex conjugate of $a(n,z)$.

Let

$$\phi=\operatorname{arccot} z.$$

Then

$$a(n,z)=-(n+\tfrac{1}{2})e^{-i\phi}.$$

In computing $\{f_n(z;\zeta),g_n(z;\zeta)\}$ it is useful to note that

$$\begin{bmatrix}f_n(z;\zeta)\\ g_n(z;\zeta)\end{bmatrix}=\begin{pmatrix}A(n,z,R) & B(n,z,R)\\ -\overline{B(n,z,R)} & \overline{A(n,z,R)}\end{pmatrix}\begin{bmatrix}f_0(z;\zeta)\\ g_0(z;\zeta)\end{bmatrix} \qquad (4.17)$$

for functions $A(n,z,R)$ and $B(n,z,R)$ determined from (4.16), $n=1,2,\ldots$. Further details on $\{f_n(z;\zeta), g_n(z;\zeta)\}$ are given in the Appendix.

Say λ is real. Then $f=f_\lambda(z;\zeta)$ solves (4.2) for $\sigma=-i\lambda$ if and only if the complex conjugate of $f_\lambda(z;\zeta)$, namely $f=\overline{f_\lambda(z;\zeta)}$ solves (4.2) for $\sigma=i\lambda$. Consequently, for $\sigma=in$, $n=1,2,\ldots$, we have closed form solutions

$$f=f_{-n}(z;\zeta)=\overline{f_n(z;\zeta)},$$
$$g=g_{-n}(z;\zeta)=\overline{g_n(z;\zeta)},$$

for system (3.20).

Moreover if λ is real, using (3.20), one can prove that

$$|f_\lambda(z;\zeta)|^2 + |g_\lambda(z;\zeta)|^2 = \frac{\text{const}}{1+z^2} = \frac{|f_\lambda(0;\zeta)|^2 + |g_\lambda(0;\zeta)|^2}{1+z^2}. \tag{4.18}$$

Other sequences of solutions are found by the standard technique of using the raising and lowering operators (4.9) and (4.10). Namely we first find functions f which satisfy (4.2) and $L^-(\lambda)f=0$ or $L^+(\lambda)f=0$ for particular values of $\lambda = i\sigma$. For the lowering operator $L^-(\lambda)$, resulting solutions are

$$f=\tilde{f}_0^\pm = (1/\sqrt{z^2+1})e^{(i/2)[\arctan z \mp R\log(z^2+1)]} \tag{4.19}$$

for $\lambda = \frac{1}{2} \pm iR$. For the raising operator $L^+(\lambda)$, solutions are

$$f=\hat{f}_0^\pm = (1/\sqrt{z^2+1})e^{-(i/2)[\arctan z \pm R\log(z^2+1)]} \tag{4.20}$$

for $\lambda = -\frac{1}{2} \pm iR$. Sequences of solutions $\{\tilde{f}_0^\pm, \tilde{f}_1^\pm, \tilde{f}_2^\pm,...\}$, $\{\hat{f}_0^\pm, \hat{f}_{-1}^\pm, \hat{f}_{-2}^\pm,...\}$, are then obtained as follows:

$$\tilde{f}_{n+1}^\pm = L^+(n+\tfrac{1}{2} \pm iR)\tilde{f}_n^\pm \tag{4.21}$$

for $\lambda = n + \frac{3}{2} \pm iR$, $n = 0,1,2,...$, and

$$\hat{f}_{n-1}^\pm = L^-(n-\tfrac{1}{2} \pm iR)\hat{f}_n^\pm \tag{4.22}$$

for $\lambda = n - \frac{3}{2} \pm iR$, $n = 0, -1, -2,...$.

C. Properties of solutions (3.14) and (3.15)

Since the solutions (3.14) and (3.15) depend on similarity variable z we examine the similarity curves

$$z = \text{const} = \beta(t)\sin y = \beta(t)\sin[\nu \log c(x)], \tag{4.23}$$

where

$$\beta(t) = pe^t - qe^{-t}, \quad pq = \tfrac{1}{4}. \tag{4.24}$$

We consider solutions for $t \in (-\infty, \infty)$. Then without loss of generality we can set $p = q = \frac{1}{2}$ by a suitable choice of initial time t, so that

$$\beta(t) = \sinh t. \tag{4.25}$$

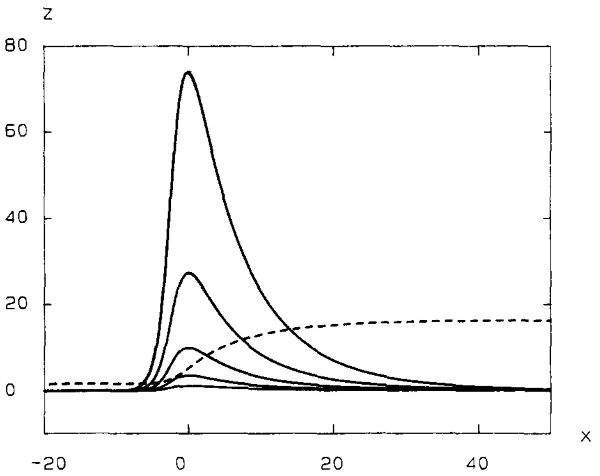

FIG. 4. z as a function of x; z is plotted as a function of x for $\nu = 1.4$ and selected values of t: $t = 1$ (top), 2, 3, 4, 5 (bottom).

Representative similarity curves are plotted in Fig. 3 for various value of z for $\nu = 1.4$. For various values of t, curves

$$z(x,t) = \sinh t \sin[\nu \log c(x)] \tag{4.26}$$

are plotted in Fig. 4 for $\nu = 1.4$. Note that $z(0,t) = \sinh t$, $\lim_{x\to\pm\infty} z(x,t) = 0$ and hence for fixed t, the range of $z(x,t)$ is $(0, \sinh t]$ if $t>0$ and $[\sinh t, 0)$ if $t<0$.

Consider the asymptotic properties of the similarity curves

$$z = \sinh t \sin[\nu \log c(x)] = \text{const} \quad \text{as } t \to +\infty.$$

From (2.11) and (2.12), along such curves we have

$$x \sim -(t - \log[2z/\nu C^-(\nu)]) \quad \text{if } x<0;$$

$$x \sim e^{\pi/\nu}(t - \log[2z/\nu C^+(\nu)]) \quad \text{if } x>0.$$

Hence as $t \to +\infty$ the similarity curves are asymptotic to the characteristic curves of the wave equation (1.1) or system (1.2). For comparison with the similarity curves of Fig. 3, characteristic curves are plotted in Fig. 5 for $\nu = 1.4$.

Next we consider properties of $\{f(z), g(z)\}$. First of all note that $f(z)$ and $g(z)$ are analytic in z. For any σ, as

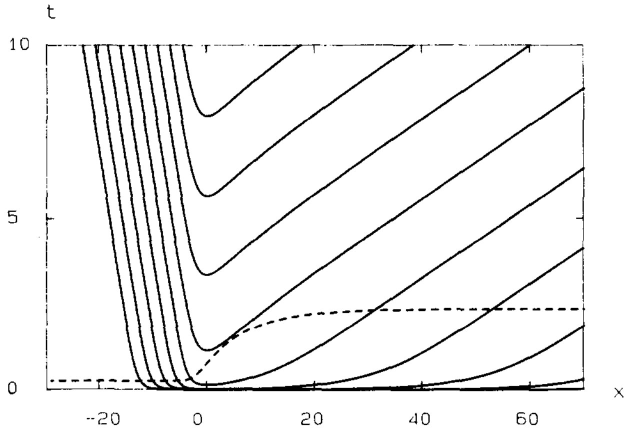

FIG. 3. Similarity curves $z = (\sinh t)\sin(\nu \log c(x))$. Nine similarity curves are plotted for $\nu = 1.4$. The corresponding values of z are $z = 10^n$ with $n = 3$ (top line), 2, 1, 0, -1, -2, -3, -4, -5 (bottom line). The dashed line represents the profile of $c(x)$ for $\nu = 1.4$.

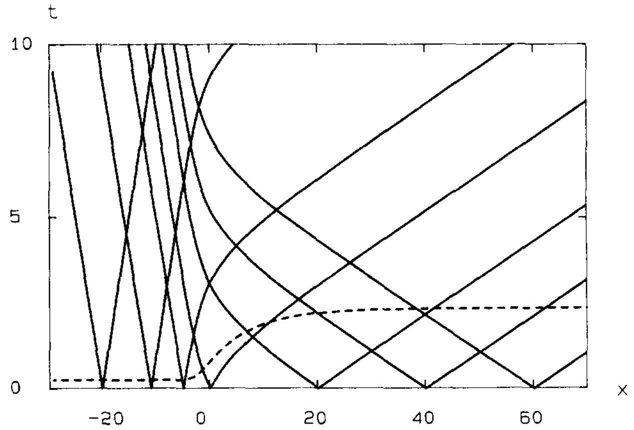

FIG. 5. Characteristic curves, defined by $dx/dt = \pm c(x)$, emanating from the x axis, are plotted for $\nu = 1.4$.

$|z|\to\infty$, from (3.20), $\{f(z),g(z)\}$ satisfy

$$z\frac{df}{dz}+f-Rg=0,\quad z\frac{df}{dz}+Rf+g=0. \tag{4.27}$$

Thus

$$\begin{aligned}f(z)&\sim(\mu^{\pm}/z)\cos[R\log|z|+\rho^{\pm}],\\ g(z)&\sim-(\mu^{\pm}/z)\sin[R\log|z|+\rho^{\pm}],\end{aligned} \tag{4.28}$$

as $z\to\pm\infty$ for some constants $\{\mu^+,\mu^-,\rho^+,\rho^-\}$. Thus $\{f(z),g(z)\}$ exhibit oscillatory algebraic decay as $|z|\to\infty$. In the Appendix, μ^+ and ρ^+ are computed for $\{f_n(z;\zeta),g_n(z;\zeta)\}$.

Now consider properties of $\epsilon(x,t)$ defined by Eq. (3.17) with $\beta(t)=\sinh t$, i.e.,

$$\epsilon(x,t)=(1/2\nu)\arctan[\operatorname{sech} t\cot y]. \tag{4.29}$$

The range of $\epsilon(x,t)$ is $(-\pi/4\nu,\pi/4\nu)$ for any t. If $\sigma=-s/2\nu=\sigma_1+i\sigma_2$, then for any $t=t^*$ the number of oscillations with respect to x in a real solution $\{u(x,t^*),v(x,t^*)\}$ due to the factor $e^{-s\epsilon(x,t^*)}$ is the integer n such that

$$n\leqslant\tfrac{1}{2}|\sigma_2|<n+1, \tag{4.30}$$

and for any $x=x^*$ the number of oscillations with respect to t in a real solution $\{u(x^*,t),\ v(x^*,t)\}$ due to the factor $e^{-s\epsilon(x^*,t)}$ is the integer m such that

$$m\leqslant\tfrac{1}{2}|\sigma_2|\cdot|\tfrac{1}{2}-(\nu/\pi)\log c(x^*)|<m+1. \tag{4.31}$$

Furthermore,

$$\epsilon(x,0)=\frac{1}{2\nu}\left[\frac{\pi}{2}-y\right]=\frac{1}{2\nu}\left[\frac{\pi}{2}-\nu\log c(x)\right], \tag{4.32}$$

$$\epsilon(0,t)=0, \tag{4.33}$$

$$\lim_{t\to\pm\infty}\epsilon(x,t)=0, \tag{4.34}$$

and

$$\lim_{x\to\pm\infty}\epsilon(x,t)=\mp(\pi/4\nu). \tag{4.35}$$

At $t=0$,

$$\begin{aligned}u(x,0)&=e^{-s\epsilon(x,0)}[c(x)\sin[\nu\log c(x)]]^{1/2}\\&\quad\times[f(0)+g(0)],\\ v(x,0)&=e^{-s\epsilon(x,0)}[c(x)]^{-1/2}[\sin[\nu\log c(x)]]^{1/2}\\&\quad\times[f(0)-g(0)],\end{aligned} \tag{4.36}$$

where $\epsilon(x,0)$ is given by (4.32). In both the real and imaginary parts of (4.36) the number of oscillations with respect to x is the integer n given by (4.30).

At $x=0$,

$$\begin{aligned}u(0,t)&=e^{\pi/4\nu}\sqrt{\cosh t}\,[f(\sinh t)+g(\sinh t)],\\ v(0,t)&=e^{-\pi/4\nu}\sqrt{\cosh t}\,[f(\sinh t)-g(\sinh t)].\end{aligned} \tag{4.37}$$

Thus $\{u(0,t),\ v(0,t)\}$ are finite in t.

Moreover,

$$\lim_{x\to\pm\infty}u(x,t)=\lim_{x\to\pm\infty}v(x,t)=0. \tag{4.38}$$

Let

$$\Theta(x,t)=R\,[|t|+\log[\tfrac{1}{2}\sin y]]. \tag{4.39}$$

Then as $t\to+\infty$, x fixed,

$$\begin{aligned}u(x,t)&\sim2\mu^+[c(x)/\sin y]^{1/2}e^{-t/2}\\&\quad\times\cos[\Theta(x,t)+\tfrac{1}{2}y+\rho^+],\\ v(x,t)&\sim2\mu^+[c(x)\sin y]^{-1/2}e^{-t/2}\\&\quad\times\cos[\Theta(x,t)-\tfrac{1}{2}y+\rho^+].\end{aligned} \tag{4.40}$$

As $t\to-\infty$, x fixed,

$$\begin{aligned}u(x,t)&\sim2\mu^-[c(x)/\sin y]^{1/2}e^{t/2}\\&\quad\times\sin[\Theta(x,t)-\tfrac{1}{2}y+\rho^-],\\ v(x,t)&\sim-2\mu^-[c(x)\sin y]^{-1/2}e^{t/2}\\&\quad\times\sin[\Theta(x,t)+\tfrac{1}{2}y+\rho^-].\end{aligned} \tag{4.41}$$

More importantly as $t\to+\infty$ along a similarity curve $z=$ const, one can show that if $x<0$, then

$$\begin{aligned}u(x,t)&=\sqrt{2z}\,e^{\sigma\arctan(1/z)}f(z)\,[1+(z/\nu)e^{-t}+o(e^{-t})],\\ v(x,t)&=\sqrt{2z}\,e^{\sigma\arctan(1/z)}f(z)\,[1-(z/\nu)e^{-t}+o(e^{-t})];\end{aligned} \tag{4.42}$$

if $x>0$, then

$$\begin{aligned}u(x,t)&=\sqrt{2z}\,e^{\pi/2\nu}e^{-\sigma\arctan(1/z)}\\&\quad\times g(z)\,[1-(z/\nu)e^{-t}+o(e^{-t})],\\ v(x,t)&=-\sqrt{2z}\,e^{-\pi/2\nu}e^{-\sigma\arctan(1/z)}\\&\quad\times g(z)\,[1+(z/\nu)e^{-t}+o(e^{-t})].\end{aligned} \tag{4.43}$$

V. SUPERPOSITION OF INVARIANT SOLUTIONS; SOLUTION OF THE INITIAL VALUE PROBLEM

By superposing invariant solutions, general initial value problems (IVP's) of the form

$$u(x,0)=U(x),\quad v(x,0)=V(x),\quad -\infty<x<\infty, \tag{5.1}$$

for system (1.2), and

$$u(x,0)=U(x),\quad u_t(x,0)=W(x),\quad -\infty<x<\infty, \tag{5.2}$$

for Eq. (1.1), can be solved. Solutions $u(x,t)$ of (1.1) and (1.2) are identical if

$$W(x)=c^2(x)V'(x). \tag{5.3}$$

For $\sigma=-2mi$, i.e., $s=4\nu mi$, $m=0,\pm1,\pm2,\ldots$, consider invariant solutions (3.14) and (3.15) of system (1.2) $u=u_m(x,t;\zeta_{2m})$, $v=v_m(x,t;\zeta_{2m})$,

$$\begin{aligned}&u_m(x,t;\zeta_{2m})\\&\quad=\exp(-i2m\arctan[\cot y\operatorname{sech} t])\cdot[c(x)\sin y]^{1/2}\\&\qquad\times\{[\cosh t+\sinh t\cos y]^{1/2}f_{2m}(z;\zeta_{2m})\\&\qquad+[\cosh t-\sinh t\cos y]^{1/2}g_{2m}(z;\zeta_{2m})\},\end{aligned} \tag{5.4}$$

$$\begin{aligned}v_m(x,t;\zeta_{2m})&=\exp(-i2m\arctan[\cot y\operatorname{sech} t])\cdot\left[\frac{\sin y}{c(x)}\right]^{1/2}\\&\quad\times\{[\cosh t+\sinh t\cos y]^{1/2}f_{2m}(z;\zeta_{2m})\\&\quad-[\cosh t-\sinh t\cos y]^{1/2}g_{2m}(z;\zeta_{2m})\},\end{aligned} \tag{5.5}$$

where $\{f_{2m}(z;\zeta), g_{2m}(z;\zeta)\}$ are defined by (4.7), (4.8), and (4.16).

At $t=0$,

$$u_m(x,0;\zeta_{2m}) = (-1)^m[c(x)\sin y]^{1/2}[f_{2m}(0;\zeta_{2m}) + g_{2m}(0;\zeta_{2m})]e^{i2my}, \quad (5.6)$$

$$v_m(x,0;\zeta_{2m}) = (-1)^m[\sin y/c(x)]^{1/2}[f_{2m}(0;\zeta_{2m}) - g_{2m}(0;\zeta_{2m})]e^{i2my}. \quad (5.7)$$

For solving an initial value problem it is necessary that $\zeta_{-2m} = \zeta_{2m}$. Note that $0<2y<2\pi$. We let a superposition of invariant solutions,

$$u(x,t) = \sum_{m=-\infty}^{\infty} A_m u_m(x,t;\zeta_{2m}),$$
$$v(x,t) = \sum_{m=-\infty}^{\infty} A_m v_m(x,t;\zeta_{2m}), \quad (5.8)$$

represent the solution of the initial value problem (5.1) for system (1.2). The constants $\{A_m, \zeta_{2m}\}$ are to be determined. In practice we determine $\{A_m \cos\zeta_{2m}, A_m \sin\zeta_{2m}\}$ due to the form of (5.8). Clearly $A_{-m} = \overline{A}_m$ since $u(x,t)$ and $v(x,t)$ are real.

The initial condition (5.1) and (5.6)–(5.8) lead to the following Fourier series representations:

$$U(x)[c(x)\sin y]^{-1/2} = \sum_{m=-\infty}^{\infty} B_m e^{i2my},$$
$$V(x)\left[\frac{c(x)}{\sin y}\right]^{1/2} = \sum_{m=-\infty}^{\infty} C_m e^{i2my}, \quad (5.9)$$

where

$$B_m = (-1)^m[f_{2m}(0;\zeta_{2m}) + g_{2m}(0;\zeta_{2m})]A_m,$$
$$C_m = (-1)^m[f_{2m}(0;\zeta_m) - g_{2m}(0;\zeta_{2m})]A_m, \quad (5.10)$$
$$m = 0, \pm 1, \pm 2, \ldots,$$

$$B_m = \frac{1}{\pi}\int_0^\pi e^{-i2my}U(x(y))e^{-y/2\nu}[\sin y]^{-1/2}\,dy,$$
$$C_m = \frac{1}{\pi}\int_0^\pi e^{-i2my}V(x(y))e^{y/2\nu}[\sin y]^{-1/2}\,dy, \quad (5.11)$$

where x and y are related in a 1:1 manner by $y = \nu \log c(x)$. This completes the solution of the IVP (5.1) of system (1.2). The convergence properties of the Fourier series (5.9) depend on the nature of the functions $U(x)[c(x)\sin y]^{-1/2}$, $V(x)[c(x)/\sin y]^{1/2}$. If $\lim_{x\to\pm\infty} U(x) = \lim_{x\to+\infty} V(x) = 0$, $U(x)$, $V(x)$ bounded on $(-\infty,\infty)$ then the series (5.9) converge in the mean.

See the Appendix for general expressions for $\{f_{2m}(0;\zeta) \pm g_{2m}(0;\zeta)\}$ and discussion of the algorithm to compute (5.8).

Now we give the algorithm to find the Green's functions $(G_i(x,\xi,t), K_i(x,\xi,t))$, $i=1,2$, for the initial value problem (5.1). Here $(u,v) = (G_i(x,\xi,t), K_i(x,\xi,t))$, $i=1,2$, satisfies (1.2), and

$$G_1(x,\xi,0) = \delta(x-\xi), \quad K_1(x,\xi,0) = 0;$$
$$G_2(x,\xi,0) = 0, \quad K_2(x,\xi,0) = \delta(x-\xi). \quad (5.12)$$

In terms of these Green's functions, the solution of the IVP (5.1) for system (1.2) may be formally represented as

$$u = \int_{-\infty}^{\infty} [G_1(x,\xi,t)U(\xi) + G_2(x,\xi,t)V(\xi)]d\xi,$$
$$v = \int_{-\infty}^{\infty} [K_1(x,\xi,t)U(\xi) + K_2(x,\xi,t)V(\xi)]d\xi. \quad (5.13)$$

In computing the coefficients for (G_i, K_i) we set $C_m = C^i_m$, $B_m = B^i_m$, $A_m = A^i_m$, $\zeta_{2m} = \zeta^i_{2m}$, $i=1,2$. Then (5.11) gives

$$C^1_m = 0,$$
$$B^1_m = \frac{1}{\pi}[c(\xi)]^{-3/2}[\sin[\nu\log c(\xi)]]^{1/2}e^{-i2m\nu\log c(\xi)},$$
$$C^2_m = c(\xi)B^1_m,$$
$$B^2_m = 0. \quad (5.14)$$

Now from (5.10), (5.14), (5.8), (5.4), and (5.5) it follows that $\{G_1, K_1, G_2, K_2\}$ are of the form

$$G_1(x,\xi,t) = [c(\xi)]^{-3/2}[\sin[\nu\log c(\xi)]^{1/2}] \times \sum_{m=-\infty}^{\infty} a^1_m e^{-i2m\nu\log c(\xi)}U^1_m(x,t),$$
$$K_1(x,\xi,t) = [c(\xi)]^{-3/2}[\sin[\nu\log c(\xi)]]^{1/2} \times \sum_{m=-\infty}^{\infty} b^1_m e^{-i2m\nu\log c(\xi)}V^1_m(x,t), \quad (5.15)$$
$$G_2(x,\xi,t) = [c(\xi)]^{-1/2}[\sin[\nu\log c(\xi)]]^{1/2} \times \sum_{m=-\infty}^{\infty} a^2_m e^{-i2m\nu\log c(\xi)}U^2_m(x,t),$$
$$K_2(x,\xi,t) = [c(\xi)]^{-1/2}[\sin[\nu\log c(\xi)]]^{1/2} \times \sum_{m=-\infty}^{\infty} b^2_m e^{-i2m\nu\log c(\xi)}V^2_m(x,t), \quad (5.15)$$

where the constants $\{a^i_m, b^i_m\}$ and the functions $\{U^i_m(x,t), V^i_m(x,t)\}$, $i=1,2$, are independent of ξ.

Now consider (5.11) for hump functions (unimodal functions)

$$U(x) = (\sin y)^{n+1/2}e^{(1/2)\alpha y}, \quad V(x) = 0, \quad (5.16)$$

where $n = 0,1,2,\ldots$, and α is an arbitrary real constant.

Then $\lim_{x\to\pm\infty} U(x) = 0$, and $U(x)$ has precisely one extremum (a maximum) located at $y = y^\dagger$, $0<y^\dagger<\pi$, where

$$y^\dagger = \operatorname{arccot}(-\alpha/(2n+1)). \quad (5.17)$$

Let

$$\kappa = \alpha/(2n+1), \quad (5.18)$$

$$U(x;\kappa,n) = [\sin y e^{\kappa y}/\sin y^\dagger e^{\kappa y^\dagger}]^{n+1/2}, \quad n = 0,1,2,\ldots. \quad (5.19)$$

For each n, the hump function $U(x) = U(x;\kappa,n)$ has amplitude 1 with its maximum located at $y = y^\dagger = \operatorname{arccot}(-\kappa)$.

If $y^\dagger$ is fixed and n increases, from (5.19) it follows that the hump sharpens. It sharpens to a spike as $n\to\infty$. Three profiles of $U(x)$ are plotted in Figs. 6(a) and 6(b) for $n=0$ and $n=10$, respectively, with $\nu = 1.4$.

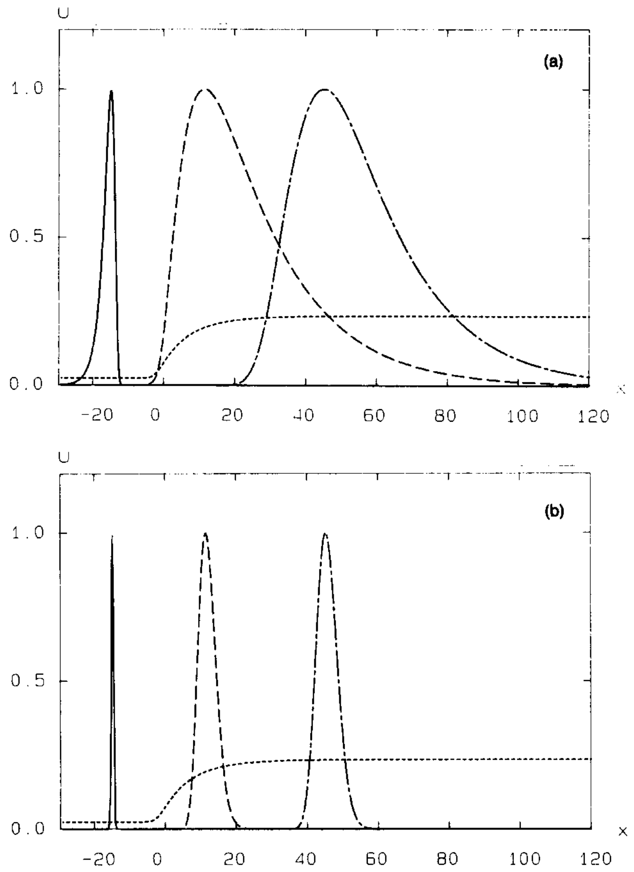

FIG. 6. (a) Hump function $U(x;\kappa,0)$; (b) Hump function $U(x;\kappa,10)$. Three hump functions are plotted for $n=0$ [Fig. (a)] and $n=10$ [Fig. (b)]. In both cases the locations of the peaks are at $x=-15$, 11.25, and 45 and the corresponding values of κ are about -2.25×10^5, 3.73, and 1.62×10^2, respectively. The value of ν is 1.4.

Let

$$A(\kappa,n)=[\sin y^\dagger e^{\kappa y^\dagger}]^{-(n+1/2)}. \tag{5.20}$$

Corresponding to $U(x;\kappa,n)$,

$$B_m=B_m(\kappa,n)=\frac{A(\kappa,n)}{\pi}\int_0^\pi e^{-i2my}e^{b(\kappa,n)y}\sin^n y\,dy,$$

which integrates to

$$B_m=n!\,[A(\kappa,n)/a\pi]\times\frac{(e^{b\pi}-1)}{(a^2+n^2)(a^2+(n-2)^2)\cdots(a^2+2^2)}$$

$$\text{if } n=2N,\quad N=1,2,\ldots,$$

$$B_m=n!\,[A(\kappa,n)/\pi]\times\frac{(e^{b\pi}+1)}{(a^2+n^2)(a^2+(n-2)^2)\cdots(a^2+1^2)}$$

$$\text{if } n=2N-1,\quad N=1,2,\ldots, \tag{5.21}$$

with

$$b=b(\kappa,n)=\tfrac{1}{2}[(2n+1)\kappa-1/\nu], \tag{5.22}$$

and

$$a=a(\kappa,m,n)=b(\kappa,n)-2mi. \tag{5.23}$$

One can show that

$$B_m(\kappa,2N)=\frac{(2N)!\,A(\kappa,2N)}{\pi}(e^{b\pi}-1)\frac{b+2mi}{b^2+4m^2}\times\prod_{k=1}^{N}\frac{b^2+4k^2-4m^2+4bmi}{(b^2+4k^2-4m^2)^2+16b^2m^2},$$

$$B_m(\kappa,2N-1)=\frac{(2N-1)!\,A(\kappa,2N-1)}{\pi}(e^{b\pi}+1)\times\prod_{k=1}^{N}\frac{b^2+(2k-1)^2-4m^2+4bmi}{(b^2+(2k-1)^2-4m^2)^2+16b^2m^2}, \tag{5.24}$$

$$N=1,2,\ldots.$$

If $n=0$,

$$B_m(\kappa,0)=(2/\pi)A(\kappa,0)[e^{(1/2)(\kappa-1/\nu)\pi}-1]\times\left[\frac{(\kappa-1/\nu)+4mi}{(\kappa-1/\nu)^2+16m^2}\right]. \tag{5.25}$$

ACKNOWLEDGMENT

This work was supported in part by the Natural Sciences and Engineering Council of Canada under Grant No. A7157.

APPENDIX

1. Computation of $\{f_{2m}(0;\zeta)\pm g_{2m}(0;\zeta)\}$

From (4.16), it follows that

$$\begin{bmatrix}f_{n+1}(0;\zeta)\\ g_{n+1}(0;\zeta)\end{bmatrix}=\begin{pmatrix}i(n+\frac12) & R\\ -R & -i(n+\frac12)\end{pmatrix}\begin{bmatrix}f_n(0;\zeta)\\ g_n(0;\zeta)\end{bmatrix},$$

$$n=0,1,2,\ldots. \tag{A1}$$

Hence

$$\begin{bmatrix}f_{n+1}(0;\zeta)+g_{n+1}(0;\zeta)\\ f_{n+1}(0;\zeta)-g_{n+1}(0;\zeta)\end{bmatrix}=\begin{pmatrix}0 & i(n+\frac12)-R\\ i(n+\frac12)+R & 0\end{pmatrix}\begin{bmatrix}f_n(0;\zeta)+g_n(0;\zeta)\\ f_n(0;\zeta)-g_n(0;\zeta)\end{bmatrix}, \tag{A2}$$

$$n=0,1,2,\ldots.$$

It follows that

$$\begin{bmatrix}f_{n+2}(0;\zeta)+g_{n+2}(0;\zeta)\\ f_{n+2}(0;\zeta)-g_{n+2}(0;\zeta)\end{bmatrix}=-\begin{pmatrix}(n+\frac32)(n+\frac12)+R^2-iR & 0\\ 0 & (n+\frac32)(n+\frac12)+R^2+iR\end{pmatrix}\begin{bmatrix}f_n(0;\zeta)+g_n(0;\zeta)\\ f_n(0;\zeta)-g_n(0;\zeta)\end{bmatrix},$$

$$n=0,1,2,\ldots. \tag{A3}$$

Let

$$\alpha_m = \arctan \frac{4R}{(4m-1)(4m-3)+4R^2}, \tag{A4}$$

$$s_m = ([(4m-1)(4m-3)+4R^2]^2 + 16R^2)^{1/2}, \tag{A5}$$

and

$$\Theta_m = \alpha_1 + \alpha_2 + \cdots + \alpha_m, \quad m = 1,2,\ldots . \tag{A6}$$

Then

$$f_{2m}(0;\zeta) \pm g_{2m}(0;\zeta) = [(-1)^m/4^m](s_1 s_2 \cdots s_m) e^{\mp i\Theta_m}[\cos\zeta \mp \sin\zeta], \quad m = 1,2,\ldots . \tag{A7}$$

Note that

$$\begin{aligned} f_{2m}(0;\zeta) &= [(-1)^m/4^m](s_1 s_2 \cdots s_m) \\ &\quad \times [\cos\Theta_m \cos\zeta + i \sin\Theta_m \sin\zeta], \\ g_{2m}(0;\zeta) &= [(-1)^{m+1}/4^m](s_1 s_2 \cdots s_m) \\ &\quad \times [\cos\Theta_m \sin\zeta + i \sin\Theta_m \cos\zeta], \\ &\quad m = 1,2,\ldots . \end{aligned} \tag{A8}$$

Thus [cf. (4.18)]

$$(|f_{2m}(0;\zeta)|^2 + |g_{2m}(0;\zeta)|^2)^{1/2} = s_1 s_2 \cdots s_m/4^m, \quad m = 1,2,\ldots . \tag{A9}$$

2. Computation of $\{f_n(z;\zeta), g_n(z;\zeta)\}$

Consider (4.16). The matrix

$$M(n,z,R) \equiv \frac{1}{\sqrt{(n+\frac{1}{2})^2+R^2}} \begin{pmatrix} a(n,z) & R \\ -R & \overline{a(n,z)} \end{pmatrix} \tag{A10}$$

is a unitary matrix.

Let

$$\beta_n = \arctan[2R/(2n+1)], \quad n = 0,1,2,\ldots, \tag{A11}$$

and

$$f = \operatorname{Re} f + i \operatorname{Im} f.$$

Then

$$\begin{bmatrix} \operatorname{Re} f_{n+1}(z;\zeta) \\ \operatorname{Im} f_{n+1}(z;\zeta) \\ \operatorname{Re} g_{n+1}(z;\zeta) \\ \operatorname{Im} g_{n+1}(z;\zeta) \end{bmatrix} = -\sqrt{(n+\tfrac{1}{2})^2+R^2}\, N(\beta_m,\phi) \begin{bmatrix} \operatorname{Re} f_n(z;\zeta) \\ \operatorname{Im} f_n(z;\zeta) \\ \operatorname{Re} g_n(z;\zeta) \\ \operatorname{Im} g_n(z;\zeta) \end{bmatrix}, \tag{A12}$$

where $\phi = \operatorname{arccot} z$, and the 4×4 orthogonal matrix

$$N(\beta_n,\phi) \equiv \begin{pmatrix} \cos\beta_n \cos\phi & \cos\beta_n \sin\phi & -\sin\beta_n & 0 \\ -\cos\beta_n \sin\phi & \cos\beta_n \cos\phi & 0 & -\sin\beta_n \\ \sin\beta_n & 0 & \cos\beta_n \cos\phi & -\cos\beta_n \sin\phi \\ 0 & \sin\beta_n & \cos\beta_n \sin\phi & \cos\beta_n \cos\phi \end{pmatrix}, \quad n = 0,1,2,\ldots, \tag{A13}$$

and

$$\begin{bmatrix} \operatorname{Re} f_0(z;\zeta) \\ \operatorname{Im} f_0(z;\zeta) \\ \operatorname{Re} g_0(z;\zeta) \\ \operatorname{Im} g_0(z;\zeta) \end{bmatrix} = \frac{1}{\sqrt{z^2+1}} \begin{bmatrix} \cos[R \log(z+\sqrt{z^2+1})+\zeta] \\ 0 \\ -\sin[R \log(z+\sqrt{z^2+1})+\zeta] \\ 0 \end{bmatrix}. \tag{A14}$$

3. Asymptotic properties of $\{f_n(z;\zeta), g_n(z;\zeta)\}$

As $z \to +\infty$, from (4.16),

$$\begin{bmatrix} f_{n+1}(z;\zeta) \\ g_{n+1}(z;\zeta) \end{bmatrix} \sim -\sqrt{(n+\tfrac{1}{2})^2+R^2} \times \begin{pmatrix} \cos\beta_n & -\sin\beta_n \\ \sin\beta_n & \cos\beta_n \end{pmatrix} \begin{bmatrix} f_n(z;\zeta) \\ g_n(z;\zeta) \end{bmatrix}, \quad n = 0,1,2,\ldots . \tag{A15}$$

From (A15) and an analysis of the error in (A15), one can show that as $z \to +\infty$ [cf. (4.28)],

$$\begin{aligned} f_n(z;\zeta) &= (\mu_n^+/z)[\cos[R \log z + \rho_n^+]] \\ &\quad \times [1+O(1/z)], \\ g_n(z;\zeta) &= -(\mu_n^+/z)\sin[[R \log z + \rho_n^+]] \\ &\quad \times [1+O(1/z)], \end{aligned} \tag{A16}$$

where

$$\mu_n^+ = (-1)^n([(n-\tfrac{1}{2})^2+R^2][(n-\tfrac{3}{2})^2+R^2] \times \cdots \times [(\tfrac{1}{2})^2+R^2])^{1/2}, \tag{A17}$$

$$\rho_n^+ = R\log 2 + \zeta - \sum_{k=0}^{n-1}\beta_k, \quad n = 1,2,...; \tag{A18}$$

$\{\beta_k\}$ defined by (A11).

4. Discussion of the algorithm to compute (5.8)

For $n = 2m$, consider the matrix defined by (4.17), namely

$$P_{2m}(z,R) = \begin{pmatrix} A(2m,z,R) & B(2m,z,R) \\ -\overline{B(2m,z,R)} & \overline{A(2m,z,R)} \end{pmatrix}. \tag{A19}$$

Then

$$\begin{bmatrix} f_{2m}(z;\zeta_{2m}) \\ g_{2m}(z;\zeta_{2m}) \end{bmatrix} = P_{2m}(z,R)\begin{bmatrix} f_0(z;\zeta_{2m}) \\ g_0(z;\zeta_{2m}) \end{bmatrix}. \tag{A20}$$

Note that the matrix $P_{2m}(z,R)$ is independent of ζ_{2m}. From (4.7) and (4.8),

$$\begin{bmatrix} f_0(z;\zeta_{2m}) \\ g_0(z;\zeta_{2m}) \end{bmatrix} = \frac{1}{\sqrt{z^2+1}}\begin{pmatrix} \cos R\Psi(z) & \sin R\Psi(z) \\ -\sin R\Psi(z) & \cos R\Psi(z) \end{pmatrix} \times \begin{bmatrix} \cos\zeta_{2m} \\ -\sin\zeta_{2m} \end{bmatrix}. \tag{A21}$$

Now multiply both sides of (A20) by A_m and set $z = 0$. Then

$$A_m\begin{bmatrix} f_{2m}(0;\zeta_{2m}) \\ g_{2m}(0;\zeta_{2m}) \end{bmatrix} = A_m P_{2m}(0,R)\begin{bmatrix} \cos\zeta_{2m} \\ -\sin\zeta_{2m} \end{bmatrix}, \tag{A22}$$

where (A7) gives

$$P_{2m}(0,R) = \frac{(-1)^m s_1 s_2\cdots s_m}{2\cdot 4^m}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} e^{-i\Theta_m} & 0 \\ 0 & e^{i\Theta_m} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{A23}$$

Thus from (A22)

$$A_m\begin{bmatrix} \cos\zeta_{2m} \\ -\sin\zeta_{2m} \end{bmatrix} = [P_{2m}(0,R)]^{-1}A_m\begin{bmatrix} f_{2m}(0;\zeta_{2m}) \\ g_{2m}(0;\zeta_{2m}) \end{bmatrix}, \tag{A24}$$

with

$$[P_{2m}(0,R)]^{-1} = \frac{(-1)^m 4^m}{2s_1 s_2\cdots s_m}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} e^{i\Theta_m} & 0 \\ 0 & e^{-i\Theta_m} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{A25}$$

From the initial condition (5.10),

$$A_m\begin{bmatrix} f_{2m}(0;\zeta_{2m}) \\ g_{2m}(0;\zeta_{2m}) \end{bmatrix} = \frac{(-1)^m}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{bmatrix} B_m \\ C_m \end{bmatrix}. \tag{A26}$$

Hence in the superposition (5.8),

$$A_m\begin{bmatrix} f_{2m}(z;\zeta_{2m}) \\ g_{2m}(z;\zeta_{2m}) \end{bmatrix} = \frac{4^m}{2s_1 s_2\cdots s_m\sqrt{z^2+1}} \times P_{2m}(z,R)\begin{pmatrix} \cos R\Psi(z) & \sin R\Psi(z) \\ -\sin R\Psi(z) & \cos R\Psi(z) \end{pmatrix} \times\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{i\Theta_m} & 0 \\ 0 & e^{-i\Theta_m} \end{pmatrix}\begin{bmatrix} B_m \\ C_m \end{bmatrix}. \tag{A27}$$

Note that explicit computations of $\{A_m,\zeta_{2m}\}$ are not required. Thus the problem of determining $\{A_m f_{2m}(z;\zeta_{2m}), A_m g_{2m}(z;\zeta_{2m})\}$ has been reduced to the computation of $P_{2m}(z,R)$.

Algebraically, $P_{2m}(z,R)$ is determined by using the recursive relation (4.16) or its real version (A12)–(A14). Next we give a nonrecursive procedure for finding $P_{2m}(z,R)$ based on a numerical solution of an initial value problem for a system of ordinary differential equations.

Let

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} F_{2m}(z) \\ G_{2m}(z) \end{bmatrix} \tag{A28}$$

solve the system corresponding to (3.20) and (4.1) for $\sigma = -2mi$, namely

$$(z^2+1)\frac{df}{dz} + (z-2mi)f - R\sqrt{z^2+1}\,g = 0,$$
$$(z^2+1)\frac{dg}{dz} + (z+2mi)g + R\sqrt{z^2+1}\,f = 0, \tag{A29}$$

with initial condition

$$\begin{bmatrix} f(0) \\ g(0) \end{bmatrix} = \frac{(-1)^m}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{bmatrix} b_m \\ c_m \end{bmatrix} \tag{A30}$$

for any nontrivial choice of constants $\{b_m,c_m\}$, $m = 1,2,...$. Then $\left[\begin{smallmatrix} F_{2m}(z) \\ G_{2m}(z) \end{smallmatrix}\right]$ equals the right-hand side of (A27) with $B_m = b_m$, $C_m = c_m$. Here $P_{2m}(z,R)$ is determined in terms of $\{F_{2m}(z), G_{2m}(z)\}$,

$$P_{2m}(z,R) = \frac{s_1 s_2\cdots s_m}{4^m[|B_m|^2+|C_m|^2]}\cdot\sqrt{z^2+1}\begin{pmatrix} F_{2m}(z) & -\overline{G_{2m}(z)} \\ G_{2m}(z) & \overline{F_{2m}(z)} \end{pmatrix} \times\begin{pmatrix} \overline{B}_m & \overline{C}_m \\ C_m & -B_m \end{pmatrix}\begin{pmatrix} e^{-i\Theta_m} & 0 \\ 0 & e^{+i\Theta_m} \end{pmatrix}\begin{pmatrix} \cos R\Psi(z) & \sin R\Psi(z) \\ -\sin R\Psi(z) & \cos R\Psi(z) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{A31}$$

Note that the matrix $P_{2m}(z,R)$ is independent of the choice of $\{b_m,c_m\}$. If $b_m = 1$, $c_m = 0$, then (A31) becomes

$$P_{2m}(z,R) = \frac{s_1 s_2\cdots s_m}{4^m}\sqrt{2(z^2+1)}\begin{pmatrix} F_{2m}(z) & -\overline{G_{2m}(z)} \\ G_{2m}(z) & \overline{F_{2m}(z)} \end{pmatrix}\begin{pmatrix} e^{-i\Theta_m}\cos(R\Psi(z)-\pi/4) & -e^{-i\Theta_m}\sin(R\Psi(z)-\pi/4) \\ e^{i\Theta_m}\sin(R\Psi(z)-\pi/4) & e^{i\Theta_m}\cos(R\Psi(z)-\pi/4) \end{pmatrix}. \tag{A32}$$

Any numerical procedure such as Runge–Kutta can be used to find $\{F_{2m}(z), G_{2m}(z)\}$, $m = 1,2,\ldots$.

The following asymptotic expression is useful for computing $\{A_m f_{2m}(z;\zeta_{2m}), A_m g_{2m}(z;\zeta_{2m})\}$ for large z: using (A11), (A16)–(A18), (A24)–(A26), one can show that as $z \to +\infty$,

$$A_m \begin{bmatrix} f_{2m}(z;\zeta_{2m}) \\ g_{2m}(z;\zeta_{2m}) \end{bmatrix} = \frac{\sqrt{2}}{z}\left[1 + O\left(\frac{1}{z}\right)\right] \frac{4^m \mu_{2m}^{+}}{2s_1 s_2 \cdots s_m} \times \begin{pmatrix} e^{i\Theta_m} \cos \omega_m(z) & -e^{-i\Theta_m} \sin \omega_m(z) \\ -e^{i\Theta_m} \sin \omega_m(z) & -e^{-i\Theta_m} \cos \omega_m(z) \end{pmatrix} \times \begin{bmatrix} B_m \\ C_m \end{bmatrix}, \quad \text{(A33)}$$

where

$$\omega_m(z) = R \log 2z - \frac{\pi}{4} - \sum_{k=0}^{2m-1} \beta_k, \quad \text{(A34)}$$

and $\{\Theta_m, s_m\}$ are given by (A4)–(A6).

[1]G. Bluman and S. Kumei, J. Math. Phys. 28, 307 (1987).
[2]B. Seymour and E. Varley, Stud. Appl. Math. 76, 1 (1987).
[3]M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 15.