New classes of symmetries for partial differential equations

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New classes of symmetries for partial differential equations are introduced. By writing a given partial differential equation S in a conserved form, a related system T with potentials as additional dependent variables is obtained. The Lie group of point transformations admitted by T induces a symmetry group of S. *New symmetries* may be obtained for S that are neither point nor Lie–Bäcklund symmetries. They are determined by a completely algorithmic procedure. Significant new symmetries are found for the wave equation with a variable wave speed and the nonlinear diffusion equation.

I. INTRODUCTION

In this paper we introduce new classes of symmetries for partial differential equations (PDE's). We present an algorithm to find such symmetries. In general, they are not determined by a direct application, to the given PDE, of Lie's method for finding point symmetries and Lie–Bäcklund symmetries. These new symmetries significantly extend the applicability of group analysis to differential equations.

A symmetry group of a differential equation is a group that maps solutions to other solutions of the differential equation.

Lie considered groups of point transformations depending on continuous parameters, acting on the space of independent and dependent variables of a given differential equation. Unlike the case for a discrete group, Lie showed that the continuous group of point transformations admitted by a differential equation can be found by an explicit algorithm (cf. Refs. 1–3 for recent accounts). Such a group is completely characterized in terms of its infinitesimal generators, which depend on the independent and dependent variables of the given differential equation. Lie extended his work to groups of contact transformations that act on the space of independent and dependent variables and first derivatives of the dependent variables of the given differential equation.

Noether[4] recognized the possibility of generalizing Lie's infinitesimals by allowing them to depend on derivatives of the dependent variables up to any finite order. Such generalized symmetries, commonly called Lie–Bäcklund transformations, came to fruition in Ref. 5. Lie–Bäcklund symmetries lead directly to the infinity of conservation laws arising in the study of the Korteweg–de Vries, sine–Gordon, nonlinear Schrödinger, and other nonlinear differential equations exhibiting soliton behavior and are computed by a simple extension of Lie's algorithm.[1,6,7]

In our approach we obtain new classes of symmetries by computing Lie groups of point transformations whose infinitesimals act on a different space than the space of independent variables, dependent variables, and their derivatives, of the given differential equation. In terms of the variables of the given differential equation, our new symmetries are neither point symmetries nor Lie–Bäcklund symmetries.

Our approach can be applied to a system S of PDE's with independent variables x and dependent variables u, written in a conserved form with respect to some choice of these variables. Through the conserved form we naturally introduce potentials ϕ. The resulting system T of PDE's has as its variables the independent variables x, the dependent variables u of S, plus new dependent variables ϕ. We find the Lie group G_T of point transformations, of this enlarged space of variables (x,u,ϕ), admitted by system T.

Any transformation in G_T maps solutions of T into other solutions of T and hence maps solutions of S into other solutions of S. Consequently, G_T is a symmetry group of S. A transformation in G_T is a *new symmetry for S* if the infinitesimal of the transformation, corresponding to any of the variables (x,u), depends explicitly on ϕ. We show that a *new symmetry* is neither a point symmetry nor a Lie–Bäcklund symmetry of S.

Our new symmetries are nonlocal symmetries that are realized as local (point) symmetries in the space (x,u,ϕ). Thus they can be found by Lie's algorithm.

Special types of nonlocal symmetries have been studied by other authors.[8–10] Their works give no explicit algorithms for finding nonlocal symmetries. In general, our nonlocal symmetries do not belong to the types considered by these authors.

In Sec. II we present our method for obtaining new symmetries admitted by PDE's. By way of example, we find new symmetries for the wave equation in Sec. III and the nonlinear diffusion equation in Sec. IV.

II. METHOD FOR FINDING NEW SYMMETRIES

Consider a PDE S of order m written in a conserved form,

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(x,u,\partial u,\partial^2 u,\ldots,\partial^{m-1}u) = 0, \tag{2.1}$$

with $n \geqslant 2$ independent variables $x = (x_1,x_2,\ldots,x_n)$ and a single dependent variable u; $\partial^j u$ represents all jth-order partials of u with respect to x. (For simplicity we consider a single PDE—the generalization to a system of PDE's in a conserved form is straightforward.)

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We remark that if a given PDE is not written in a conserved form, there are a number of ways of attempting to put it in a conserved form. As discussed in Sec. V, these include a change of variables (*dependent as well as independent*), an application of Noether's theorem, and combinations of the above.

Since Eq. (2.1) is in a conserved form, there is an $(n-1)$-exterior differential form F such that Eq. (2.1) can be written as $dF = 0$. It follows that there is an $(n-2)$-form Φ[11],

$$F = d\Phi. \tag{2.2}$$

In terms of components, Eq. (2.2) implies that there exist $\frac{1}{2}n(n-1)$ "potentials" Ψ_{ij}, components of an antisymmetric tensor, such that

$$F_i(x,u,\partial u,...,\partial^{m-1}u) = \sum_{i<j}(-1)^j\frac{\partial\Psi_{ij}}{\partial x_j} + \sum_{j<i}(-1)^{i-1}\frac{\partial\Psi_{ji}}{\partial x_j}, \quad i,j = 1,2,...,n. \tag{2.3}$$

Equation (2.3) is a system of PDE's with $1+\frac{1}{2}n(n-1)$ dependent variables u, Ψ_{ij} $(i<j)$. Thus (2.3) is underdetermined for $n\geqslant 3$. We can impose suitable constraints (effectively, a choice of gauge) on the potentials Ψ_{ij} to make system (2.3) into a determined system. A natural way to do this is to impose the conditions

$$\Psi_{ij} = 0, \quad |i-j|\neq 1. \tag{2.4}$$

In this case, letting

$$\phi_i = \Psi_{i,i+1}, \quad i = 1,2,...,n-1, \tag{2.5}$$

system (2.3) becomes the determined system T,

$$F_1 = \frac{\partial\phi_1}{\partial x_2},$$

$$F_\ell = (-1)^{\ell-1}\left[\frac{\partial\phi_\ell}{\partial x_{\ell+1}} + \frac{\partial\phi_{\ell-1}}{\partial x_{\ell-1}}\right], \quad 1<\ell<n,$$

$$F_n = (-1)^{n+1}\frac{\partial\phi_{n-1}}{\partial x_{n-1}}. \tag{2.6}$$

If $n = 2$, let $x_1 = x$, $x_2 = t$, $F_1 = F$, and $F_2 = -G$, so that S becomes

$$\frac{\partial F}{\partial x} - \frac{\partial G}{\partial t} = 0. \tag{2.7}$$

Let the potential $\Psi_{12} = \phi_1 = \phi$. Consequently, T is

$$\frac{\partial\phi}{\partial t} = F(x,t,u,\partial u,...,\partial^{m-1}u), \tag{2.8a}$$

$$\frac{\partial\phi}{\partial x} = G(x,t,u,\partial u,...,\partial^{m-1}u). \tag{2.8b}$$

If $n = 4$, let $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = t$, $F_1 = F$, $F_2 = G$, $F_3 = H$, and $F_4 = I$, so that S becomes

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} + \frac{\partial I}{\partial t} = 0. \tag{2.9}$$

The corresponding determined system T is

$$\frac{\partial\phi_1}{\partial y} = F(x,y,z,t,u,\partial u,...,\partial^{m-1}u), \tag{2.10a}$$

$$-\left[\frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial z}\right] = G(x,y,z,t,u,\partial u,...,\partial^{m-1}u), \tag{2.10b}$$

$$\frac{\partial\phi_2}{\partial y} + \frac{\partial\phi_3}{\partial t} = H(x,y,z,t,u,\partial u,...,\partial^{m-1}u), \tag{2.10c}$$

$$-\frac{\partial\phi_3}{\partial z} = I(x,y,z,t,u,\partial u,...,\partial^{m-1}u). \tag{2.10d}$$

Now assume that a system T admits a one-parameter (ϵ) Lie group of point transformations

$$x^* = f(x,u,\phi;\epsilon) = x + \epsilon\xi_T(x,u,\phi) + O(\epsilon^2), \tag{2.11a}$$

$$u^* = g(x,u,\phi;\epsilon) = u + \epsilon\eta_T(x,u,\phi) + O(\epsilon^2), \tag{2.11b}$$

$$\phi^* = h(x,u,\phi;\epsilon) = \phi + \epsilon\zeta_T(x,u,\phi) + O(\epsilon^2), \tag{2.11c}$$

where ξ_T, η_T, and ζ_T are the infinitesimals of x, u, and ϕ, respectively, of the group. This group maps a solution of T into another solution of T and hence induces a mapping of a solution of S into another solution of S. Thus the group (2.11) is a symmetry group of PDE S. *This one-parameter symmetry group of* PDE S *is a new symmetry group of* S *if and only if either* ξ_T *or* η_T *depends explicitly on* ϕ. A new symmetry of S is neither a point symmetry nor a Lie–Bäcklund symmetry of S since ϕ, as defined by system (2.6), appears only in derivative form. Hence this new symmetry cannot be expressed as a function of $(x,u,\partial u,...,\partial^k u)$, for any finite k. Clearly, from its form, a new symmetry of S is a nonlocal symmetry of S. We let G_T denote the group of all point transformations admitted by T.

A one-parameter Lie group of point transformations admitted by S, in terms of its given variables, is of the form

$$x^* = x + \epsilon\xi_S(x,u) + O(\epsilon^2), \tag{2.12a}$$

$$u^* = u + \epsilon\eta_S(x,u) + O(\epsilon^2). \tag{2.12b}$$

Let G_S denote the group of point transformations of the form (2.12) admitted by S. It is important to note that the transformations belonging to G_S with infinitesimals $\xi_S(x,u)$ and $\eta_S(x,u)$ may not belong to G_T in the following sense: there exist no transformations in G_T with infinitesimals $\xi_T(x,u,\phi)$, $\eta_T(x,u,\phi)$, and $\zeta_T(x,u,\phi)$ such that

$$\xi_T(x,u,\phi)\equiv\xi_S(x,u), \tag{2.13a}$$

$$\eta_T(x,u,\phi)\equiv\eta_S(x,u). \tag{2.13b}$$

Say S is a linear PDE and T is a linear system of PDE's. In this case, ξ_S and ξ_T depend only on x. Here we distinguish two types of new symmetries arising from a new symmetry in G_T with an infinitesimal $\xi_T(x)$.

(i) *A linear partial differential equation S is said to have a new symmetry of type I if it has a new symmetry for which there is no infinitesimal in* G_S *such that* $\xi_S(x)\equiv\xi_T(x)$.

(ii) *A linear partial differential equation S is said to have a new symmetry of type II if it has a new symmetry for which there is some infinitesimal in* G_S *such that* $\xi_S(x)\equiv\xi_T(x)$.

For a new symmetry of type II, the similarity variables (group invariants depending only on x) are identical to those for some symmetry in G_S. This is not the case for a new symmetry of type I.

There are many ways of expressing a given PDE S as a system. However, the symmetries of such a system may not induce nonlocal symmetries for S. For example, the "usual" way to find a system $\hat{T}$ related to S is to introduce new dependent variables $v_i = \partial u/\partial x_i$, $1 \leqslant i \leqslant n$. A point symmetry admitted by $\hat{T}$, namely,

$$x^* = x + \epsilon \hat{\xi}(x,u,v) + O(\epsilon^2), \tag{2.14a}$$

$$u^* = u + \epsilon \hat{\eta}(x,u,v) + O(\epsilon^2), \tag{2.14b}$$

$$v^* = v + \epsilon \hat{\zeta}(x,u,v) + O(\epsilon^2), \tag{2.14c}$$

always induces a local symmetry of S that is either a point symmetry or a Lie–Bäcklund symmetry of S.

III. EXAMPLES OF NEW SYMMETRIES FOR THE WAVE EQUATION

Consider the wave equation S:

$$c^2(x)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0. \tag{3.1}$$

Equation (3.1) can be expressed in a conserved form,

$$\frac{\partial F}{\partial x} - \frac{\partial G}{\partial t} = 0, \tag{3.2}$$

where

$$F = \frac{\partial u}{\partial x}, \tag{3.3a}$$

$$G = \frac{1}{c^2(x)}\frac{\partial u}{\partial t}. \tag{3.3b}$$

The associated system T is

$$\frac{\partial \phi}{\partial t} = \frac{\partial u}{\partial x}, \tag{3.4a}$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{c^2(x)}\frac{\partial u}{\partial t}. \tag{3.4b}$$

Let G_S and G_T be the Lie groups of point transformations admitted by S [Eq. (3.1)] and T [Eqs. (3.4)], respectively. These groups depend on the form of the wave speed $c(x)$ and were derived in Ref. 12. The results in that paper can be broadly summarized in terms of Theorems 1–5 following. [A prime denotes differentiation with respect to x; we exclude the case $c(x) = (\alpha x + \beta)^2$, with $\{\alpha,\beta\}$ arbitrary constants, for which G_S is an ∞-parameter group.]

Theorem 1: The wave equation (3.1) admits a four-parameter Lie group of point transformations G_S if and only if the wave speed $c(x)$ satisfies the fifth-order ODE

$$\left\{c^2\left[\frac{H'''}{2H' + H^2} + 3\,\frac{[2(H')^3 - 2HH'H'' - (H'')^2]}{[2H' + H^2]^2}\right]\right\}' = 0, \tag{3.5}$$

where

$$H = c'/c. \tag{3.6}$$

Theorem 2: G_T is a four-parameter Lie group of point transformations if and only if the wave speed $c(x)$ satisfies the fourth-order ODE

$$[cc'(c/c')'']' = 0. \tag{3.7}$$

Theorem 3: For any wave speed $c(x)$ satisfying ODE (3.7), there exists a new symmetry of the wave equation (3.1).

Theorem 4: The new symmetries of the wave equation (3.1) arising from G_T are new symmetries of type II if and only if the wave speed $c(x)$ satisfies the third-order ODE

$$(c/c')'' = 0. \tag{3.8}$$

The general solution of (3.8) is

$$c(x) = (\alpha x + \beta)^\gamma, \tag{3.9}$$

where $\{\alpha,\beta,\gamma\}$ are arbitrary constants.

Theorem 5: The new symmetries of the wave equation (3.1) arising from G_T are new symmetries of type I if and only if the wave speed $c(x)$ satisfies the ODE

$$cc'(c/c')'' = \text{const} \neq 0. \tag{3.10}$$

The following theorem was proved in Ref. 13.

Theorem 6: A wave speed $c(x)$ simultaneously satisfies (3.10) and (3.5) if and only if either

$$\sqrt{c} - \arctan \gamma\sqrt{c} = \alpha x + \beta, \tag{3.11a}$$

or

$$2\sqrt{c} + \log|(\sqrt{c} - \gamma)/(\sqrt{c} + \gamma)| = \alpha x + \beta, \tag{3.11b}$$

where $\{\alpha,\beta,\gamma\}$ are arbitrary constants.

From the above follows this corollary.

Corollary 1: Both of the groups G_T and G_S are four-parameter groups if and only if the wave speed $c(x)$ satisfies (3.8), (3.11a), or (3.11b). The family of wave speeds (3.8) yields new symmetries of type II and no new symmetries of type I. The families of wave speeds (3.11a) and (3.11b) yield new symmetries of type I and no new symmetries of type II.

The following representative examples illustrate the above theorems.

1. $c(x) = \sqrt{1 + e^x}$. In this case, G_S is a two-parameter group and G_T is a four-parameter group. Infinitesimal generators of their Lie algebras are

$$G_S\colon\ L_1 = u\frac{\partial}{\partial u}, \quad L_2 = \frac{\partial}{\partial t};$$

$$G_T\colon\ \tilde{L}_1 = u\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial \phi}, \quad \tilde{L}_2 = \frac{\partial}{\partial t},$$

$$\tilde{L}_3 = e^t\left\{2(1 + e^{-x})\frac{\partial}{\partial x} - (2e^{-x} + 1)\frac{\partial}{\partial t} + (u - \phi)\left[(1 + e^{-x})\frac{\partial}{\partial u} - e^{-x}\frac{\partial}{\partial \phi}\right]\right\},$$

$$\tilde{L}_4 = e^{-t}\left\{2(1 + e^{-x})\frac{\partial}{\partial x} + (2e^{-x} + 1)\frac{\partial}{\partial t} + (u + \phi)\left[(1 + e^{-x})\frac{\partial}{\partial u} - e^{-x}\frac{\partial}{\partial \phi}\right]\right\}.$$

The generators $\tilde{L}_3$ and $\tilde{L}_4$ are new symmetries of type I for the corresponding wave equation (3.1).

2. $c(x) = 1 - x^2$. In this case, G_T is a two-parameter group and G_S has four parameters. Infinitesimal generators of their Lie algebras are

$$G_S\colon\ L_1 = u\frac{\partial}{\partial u}, \quad L_2 = \frac{\partial}{\partial t},$$

$$L_3 = (1-x^2)\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u},$$

$$L_4 = t(1-x^2)\frac{\partial}{\partial x} + \frac{1}{2}\log\frac{|x+1|}{|x-1|}\frac{\partial}{\partial t} - xtu\frac{\partial}{\partial u};$$

$$G_T\colon\ \widetilde{L}_1 = u\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial \phi},\quad \widetilde{L}_2 = \frac{\partial}{\partial t}.$$

3. $c(x) = x$. Here both of the groups G_S and G_T have four parameters, and there is a new symmetry of type II for S. Infinitesimal generators of their Lie algebras are

$$G_S\colon\ L_1 = u\frac{\partial}{\partial u},\quad L_2 = \frac{\partial}{\partial t},\quad L_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},$$

$$L_4 = 2xt\frac{\partial}{\partial x} + 2\log|x|\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u};$$

$$G_T\colon\ \widetilde{L}_1 = u\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial \phi},\quad \widetilde{L}_2 = \frac{\partial}{\partial t},$$

$$\widetilde{L}_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},$$

$$\widetilde{L}_4 = 2xt\frac{\partial}{\partial x} + 2\log|x|\frac{\partial}{\partial t} + (tu - x\phi)\frac{\partial}{\partial u} - (x^{-1}u + t\phi)\frac{\partial}{\partial \phi}.$$

The infinitesimal generator $\widetilde{L}_4$ of G_T is a new symmetry of type II for S.

4. $2\sqrt{c} + \log|(\sqrt{c}-1)/(\sqrt{c}+1)| = x$. In this case, both of the groups G_S and G_T have four parameters, and there are new symmetries of type I for S. Infinitesimal generators of their Lie algebras are

$$G_S\colon\ L_1 = u\frac{\partial}{\partial u},\quad L_2 = \frac{\partial}{\partial t},$$

$$L_3 = e^{t/2}(c-1)^{-1/2}\times\left[c^{3/2}\frac{\partial}{\partial x} - \frac{\partial}{\partial t} + \frac{(c-1)}{2}u\frac{\partial}{\partial u}\right],$$

$$L_4 = e^{-t/2}(c-1)^{-1/2}\times\left[c^{3/2}\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{(c-1)}{2}u\frac{\partial}{\partial u}\right];$$

$$G_T\colon\ \widetilde{L}_1 = u\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial \phi},\quad \widetilde{L}_2 = \frac{\partial}{\partial t},$$

$$\widetilde{L}_3 = \frac{e^t}{c-1}\left\{4c^{3/2}\frac{\partial}{\partial x} - 2(c+1)\frac{\partial}{\partial t} + [(3c-1)u - 2c^{3/2}\phi]\frac{\partial}{\partial u} + [(3-c)\phi - 2c^{-1/2}u]\frac{\partial}{\partial \phi}\right\},$$

$$\widetilde{L}_4 = \frac{e^{-t}}{c-1}\left\{4c^{3/2}\frac{\partial}{\partial x} + 2(c+1)\frac{\partial}{\partial x} + [(3c-1)u + 2c^{3/2}\phi]\frac{\partial}{\partial u} + [(3-c)\phi + 2c^{-1/2}u]\frac{\partial}{\partial \phi}\right\}.$$

Any linear combination of $\widetilde{L}_3$ and $\widetilde{L}_4$ is a new symmetry of type I for S.

As these examples clearly demonstrate, our method enables one to discover systematically new symmetries of (3.1) that cannot be found by a direct application of Lie's algorithm to (3.1).

Ovsiannikov[14] recognized the difference between the groups admitted by an equation equivalent to (3.1) and by a corresponding system equivalent to (3.4). He made some cursory remarks about these differences and went no further.

IV. EXAMPLES OF NEW SYMMETRIES FOR THE NONLINEAR DIFFUSION EQUATION

Consider the nonlinear diffusion equation S,

$$\frac{\partial}{\partial x}\left[K(u)\frac{\partial u}{\partial x}\right] - \frac{\partial u}{\partial t} = 0. \tag{4.1}$$

As it is written, Eq. (4.1) is already in a conserved form,

$$\frac{\partial F}{\partial x} - \frac{\partial G}{\partial t} = 0, \tag{4.2}$$

where

$$F = K(u)\frac{\partial u}{\partial x}, \tag{4.3a}$$

$$G = u. \tag{4.3b}$$

The associated system T is

$$\frac{\partial \phi}{\partial t} = K(u)\frac{\partial u}{\partial x}, \tag{4.4a}$$

$$\frac{\partial \phi}{\partial x} = u. \tag{4.4b}$$

The group G_S of (4.1) depends on the form of the conductivity $K(u)$ and is derived in Refs. 2, 3, and 15. The results are summarized as follows.

1. $K(u)$ arbitrary. Equation (4.1) always admits a three-parameter group with infinitesimal generators

$$L_1 = \frac{\partial}{\partial t},\quad L_2 = \frac{\partial}{\partial x},\quad L_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}. \tag{4.5}$$

2. $K(u) = \lambda(u+\kappa)^\nu$, $\{\nu(\neq -\frac{4}{3}),\lambda,\kappa\}$ arbitrary constants. Here G_S is a four-parameter group with infinitesimal generators L_1, L_2, and L_3 given by (4.5), and

$$L_4 = x\frac{\partial}{\partial x} + \frac{2}{\nu}(u+\kappa)\frac{\partial}{\partial t}. \tag{4.6}$$

A limiting case is $K(u) = \lambda e^{\nu u}$.

3. $K(u) = \lambda(u+\kappa)^{-4/3}$, $\{\lambda,\kappa\}$ arbitrary constants. Here G_S is a five-parameter group with infinitesimal generators L_1, L_2, and L_3 given by (4.5), L_4 given by (4.6) with $\nu = -\frac{4}{3}$, and

$$L_5 = x^2\frac{\partial}{\partial x} - 3x(u+\kappa)\frac{\partial}{\partial u}. \tag{4.7}$$

The group G_T of system (4.4) also depends on the form of the conductivity $K(u)$. This group is presented here for the first time. The results are summarized as follows.

1. $K(u)$ arbitrary. Equations (4.4) always admit a four-parameter group with infinitesimal generators

$$\widetilde{L}_0=\frac{\partial}{\partial\phi},\quad \widetilde{L}_1=\frac{\partial}{\partial t},\quad \widetilde{L}_2=\frac{\partial}{\partial x},$$
$$\widetilde{L}_3=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial\phi}. \tag{4.8}$$

2. $K(u)=\lambda(u+\kappa)^\nu$, $\{\nu(\neq -2),\lambda,\kappa\}$ arbitrary constants. Here G_T is a five-parameter group with infinitesimal generators $\widetilde{L}_0$, $\widetilde{L}_1$, $\widetilde{L}_2$, and $\widetilde{L}_3$ given by (4.8), and

$$\widetilde{L}_4=x\frac{\partial}{\partial x}+\frac{2}{\nu}(u+\kappa)\frac{\partial}{\partial u}+\left(1+\frac{2}{\nu}\right)\phi\frac{\partial}{\partial\phi}. \tag{4.9}$$

3. $K(u)=\lambda(u+\kappa)^{-2}$, $\{\lambda,\kappa\}$ arbitrary constants. Here G_T is an ∞-parameter group with infinitesimal generators $\widetilde{L}_0$, $\widetilde{L}_1$, $\widetilde{L}_2$, and $\widetilde{L}_3$ given by (4.8), $\widetilde{L}_4$ given by (4.9), and

$$\widetilde{L}_2=-x\phi\frac{\partial}{\partial x}+(u+\kappa)[\phi+x(u+\kappa)]\frac{\partial}{\partial u}+2t\frac{\partial}{\partial\phi}, \tag{4.10a}$$

$$\widetilde{L}_5=-x(\phi^2+2t)\frac{\partial}{\partial x}+4t^2\frac{\partial}{\partial t}+(u+\kappa)\times[\phi^2+6t+2x\phi(u+\kappa)]\frac{\partial}{\partial u}+4t\phi\frac{\partial}{\partial\phi}, \tag{4.10b}$$

$$\widetilde{L}_\infty=\Theta(\phi,t)\frac{\partial}{\partial x}-u^2\frac{\partial\Theta(\phi,t)}{\partial\phi}\frac{\partial}{\partial u}, \tag{4.10c}$$

where $v=\Theta(\phi,t)$ is an arbitrary solution of the linear differential equation

$$\frac{\partial^2 v}{\partial\phi^2}-\frac{\partial v}{\partial t}=0. \tag{4.11}$$

4. $K(u)=\dfrac{1}{u^2+pu+q}\exp\left[r\displaystyle\int\frac{du}{u^2+pu+q}\right]$.

In this example, $\{p,q,r\}$ are arbitrary constants *not* satisfying either of the relationships

(a) $r=\pm 2,\quad p^2-4q>0,$

(b) $r=0,\quad p^2-4q=0.$

The cases (a) and (b) belong to 3.

Here G_T is a five-parameter group with infinitesimal generators $\widetilde{L}_0$, $\widetilde{L}_1$, $\widetilde{L}_2$, and $\widetilde{L}_3$ given by (4.8), and

$$\widetilde{L}_4=\phi\frac{\partial}{\partial x}+(r-p)t\frac{\partial}{\partial t}-(u^2+pu+q)\frac{\partial}{\partial u}-(qx+p\phi)\frac{\partial}{\partial\phi}. \tag{4.12}$$

A comparison of the groups G_S and G_T leads to the following theorem.

Theorem 7: The nonlinear diffusion equation (4.1) has a new symmetry, arising from G_T, if and only if

$$K(u)=\frac{1}{u^2+pu+q}\exp\left[r\int\frac{du}{u^2+pu+q}\right],$$

with arbitrary constants $\{p,q,r\}$.

Ovsiannikov[1,16] expressed the nonlinear diffusion equation (4.1) as a system,

$$v=K(u)\frac{\partial u}{\partial x}, \tag{4.13a}$$

$$\frac{\partial v}{\partial x}=\frac{\partial u}{\partial t}. \tag{4.13b}$$

Our remarks at the end of Sec. II show that a point symmetry of system (4.13) is always a local symmetry of the single equation (4.1). In particular, as Ovsiannikov found in his complete point group classification of system (4.13), the group is G_S when restricted to (x,t,u) space.

V. CONCLUDING REMARKS

(1) There are various ways of attempting to put a PDE S into a conserved form. One way is to find a change of variables $\bar{x}=X(x,u)$, $\bar{u}=U(x,u)$, if possible, so that S becomes a conserved form,

$$\sum_{i=1}^{n}\frac{\partial\bar{F}_i}{\partial\bar{x}_i}(\bar{x},\bar{u},\partial\bar{u},\ldots,\partial^{m-1}\bar{u})=0, \tag{5.1}$$

where $\partial^k\bar{u}$ denotes all k th-order partials of $\bar{u}$ with respect to $\bar{x}$.

Another way depends on S being represented as the Euler–Lagrange equation for some Lagrangian density L. Each one-parameter Lie group of point transformations that leaves the action integral invariant leads to a conserved form for S through an application of Noether's theorem.

The following two examples illustrate other ways of obtaining conserved forms.

Consider the Schrödinger equation S:

$$-\frac{\partial^2 u}{\partial x^2}+V(x)u=i\frac{\partial u}{\partial t}. \tag{5.2}$$

We can reexpress (5.2) in the form

$$\frac{\partial F}{\partial x}-\frac{\partial G}{\partial t}=0, \tag{5.3}$$

where

$$F=\omega(x)\frac{\partial u}{\partial x}-\omega'(x)u, \tag{5.4a}$$

$$G=-i\omega(x)u, \tag{5.4b}$$

with

$$V(x)=\omega''(x)/\omega(x). \tag{5.5}$$

The corresponding system T is

$$\frac{\partial\phi}{\partial t}=\omega(x)\frac{\partial u}{\partial x}-\omega'(x)u, \tag{5.6a}$$

$$\frac{\partial\phi}{\partial x}=-i\omega(x)u. \tag{5.6b}$$

In a future paper we will show that for a class of potentials $V(x)$, the group G_T of system (5.6) generates new symmetries for the Schrödinger equation (5.2).

For our second example we consider the nonlinear wave equation

$$c^2\left(x,t,\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0. \tag{5.7}$$

We differentiate (5.7) with respect to x and let $v=\partial u/\partial x$ so that (5.7) becomes the PDE S,

$$\frac{\partial}{\partial x}\left[c^2(x,t,v)\frac{\partial v}{\partial x}\right]-\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right)=0. \tag{5.8}$$

The corresponding system T is

$$\frac{\partial \phi}{\partial t} = c^2(x,t,v)\,\frac{\partial v}{\partial x}, \tag{5.9a}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial v}{\partial t}. \tag{5.9b}$$

If (v,ϕ) solves T, then $u(x,t)$, defined by

$$\frac{\partial u}{\partial x} = v, \tag{5.10a}$$

$$\frac{\partial u}{\partial t} = \phi, \tag{5.10b}$$

solves the nonlinear wave equation (5.7). Hence the symmetry group G_T is a symmetry group of (5.7). From the form of (5.10) we see that new symmetries may arise for (5.7).

(2) A new symmetry leads to invariant solutions of T, which, in turn, lead to solutions of S. If S and T are linear and the new symmetry is of type I, then these solutions cannot be obtained by applying the infinitesimal operators of G_S to the invariant solutions of S arising from G_S.

If a new symmetry arising from G_T has ξ_T depending only on x, then it can be used to solve boundary value (initial value) problems explicitly. New symmetries have been used to solve initial value problems for wave equations (3.1) for a class of wave speeds with a smooth transition.[17]

(3) Since the choice of conserved form is not necessarily unique, various new groups could be admitted by a given differential equation. For any conserved form the symmetries of the related system are computed by the standard Lie algorithm. The work presented in this paper, when combined with recent advances using symbolic manipulation to execute Lie's algorithm,[18] offers considerable promise for applying group methods to much wider classes of differential equations.

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