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# Sequences of Related Linear PDEs

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We introduce a transformation theory to generate chains of linear PDEs and related solution sequences. Our procedure depends on representing a given linear PDE in terms of a special equivalent system of two coupled linear PDEs where the auxiliary dependent variable satisfies the next PDE of a sequence. The solution of a PDE with variable coefficient depending on $n+1$ constants $\{\alpha_1, \alpha_2, ..., \alpha_{n+1}\}$ is obtained from any solution of a PDE of the same type with variable coefficient depending on $n$ constants $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ by a simple Bäcklund transformation. Each sequence contains two inclusive chains since the PDE with $n$ constants is a special case of the PDE with $n+2$ constants. We generate solutions of wave equations with wave speeds $C(x; \alpha_1, \alpha_2, ..., \alpha_n)$, Fokker–Planck equations with drifts $F(x; \alpha_1, \alpha_2, ..., \alpha_n)$, and diffusion equations with diffusivities $K(x; \alpha_1 \alpha_2, ..., \alpha_n)$, where $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ are arbitrary constants, $n = 1, 2, ...$. New explicit general solutions are obtained for a class of wave equations with wave speeds depending on three parameters. 

## 1. INTRODUCTION

In this paper we introduce and illustrate a transformation theory generating chains of linear partial differential equations (PDEs) of the form

$$p(x)\frac{\partial^2 u}{\partial x^2} + q(x)\frac{\partial u}{\partial x} + r(x)u = \frac{\partial^\mu u}{\partial t^\mu} \tag{1.1}$$

and their solutions ($\mu \geqslant 1$ is a fixed integer). The transformation linking one member of a chain and its solutions to the next member of that chain and its corresponding solutions is of Bäcklund type [1]. Closely related to our work are the generalized Bäcklund transformations introduced by Loewner [2, 3].

Our basic idea depends on writing a given PDE (1.1) in terms of an "equivalent" system which consists of a pair of linear PDEs. It is chosen so



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that an introduced auxiliary variable also satisfies an uncoupled auxiliary (linear) PDE of the form (1.1). By recycling this auxiliary PDE in the role of the given PDE we construct a third related PDE of the form (1.1). Iteration of this procedure produces an infinite sequence of PDEs of the form (1.1).

Many physically interesting equations arise in the form (1.1). Examples include:

(I) *Inhomogeneous wave equation*

$$C^2(x)\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}. \tag{1.2}$$

(II) *Fokker–Planck equation*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}(F(x)u)=\frac{\partial u}{\partial t}. \tag{1.3}$$

(III) *Diffusion equation*

$$\frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right)=\frac{\partial u}{\partial t}. \tag{1.4}$$

Our transformation theory is conveniently developed for PDEs in the canonical form

$$\frac{\partial^2 U}{\partial X^2}+R(X)U=\frac{\partial^\mu U}{\partial T^\mu}. \tag{1.5}$$

By a simple point transformation of the form

$$X=X(x), \tag{1.6a}$$

$$T=t, \tag{1.6b}$$

$$U(X,T)=g(x)u(x,t), \tag{1.6c}$$

any equation (1.1) can be transformed to (1.5). A standard calculation yields explicit expressions for $X(x)$, $g(x)$, and $R(X)$.

We outline our transformation theory for equations in the "canonical" form (1.5). Consequences for the physically interesting cases (I)–(III) are obtained by inverting the appropriate point transformations (1.6).

We construct two types of sequences for PDE (1.1). The first (and simplest) depends on writing the canonical PDE (1.5) as a system of two linear PDEs.

In (1.5) let $R_1(X) = R(X)$ and let $\phi_1(X) = \phi(X; \alpha)$ be the general solution of the Riccati equation

$$\frac{d\phi_1}{dX} + [\phi_1(X)]^2 = -R_1(X), \tag{1.7}$$

where $\alpha$ is an arbitrary constant. The system consists of two coupled linear PDEs for $U_1(X, T) = U(X, T)$ and an auxiliary variable $U_2(X, T)$:

$$\frac{\partial^{\mu-1} U_2}{\partial T^{\mu-1}} = \frac{\partial U_1}{\partial X} - \phi(X; \alpha) U_1, \tag{1.8a}$$

$$\frac{\partial U_1}{\partial T} = \frac{\partial U_2}{\partial X} + \phi(X; \alpha) U_2. \tag{1.8b}$$

The transformation (1.8) linking $U_1$ and $U_2$ is of Bäcklund type [1].

Application of the operator $\partial/\partial X + \phi_1(X)$ to Eq. (1.8a), and elimination of $U_2$ dependence by using (1.8b), yields

$$\frac{\partial^2 U_1}{\partial X^2} + R_1(X) U_1 = \frac{\partial^\mu U_1}{\partial T^\mu}. \tag{1.9}$$

Similarly applying operator $\partial/\partial X - \phi_1(X)$ to Eq. (1.8b) yields

$$\frac{\partial^2 U_2}{\partial X^2} + R_2(X) U_2 = \frac{\partial^\mu U_2}{\partial T^\mu}, \tag{1.10}$$

where

$$R_2(X) = R(X; \alpha) = \frac{d\phi(X; \alpha)}{dX} - [\phi(X; \alpha)]^2. \tag{1.11}$$

Thus the related PDE (1.10) for $U_2(X, T)$ has a variable coefficient that depends on an arbitrary constant $\alpha$. We use the reciprocal relationship of $U_1$ and $U_2$ in Eqs. (1.9) and (1.10) to recycle PDE (1.10) in the role of (1.9). Consequently $R_2(X)$ plays the role of $R_1(X)$ and $U_2(X, T)$ plays the role of $U_1(X, T)$ in Eq. (1.9). A new system results with corresponding auxiliary variable $U_3(X, T)$ and variable coefficient $R_3(X)$. This recycling can be extended indefinitely to produce an infinite sequence of variable coefficient PDEs $\{R_n(X), U_n(X, T)\}$, of form (1.5). Most importantly $R_n(X)$ depends on $n-1$ arbitrary constants:

$$R_n(X) = R(X; \alpha_1, \alpha_2, ..., \alpha_{n-1}). \tag{1.12}$$

If $\phi_1(X)$ is known then our construction of $\{R_n(X), U_n(X, T)\}$ is explicit. Moreover we show that if $\alpha_{n+1}=0$,

$$R_{n+2}(X) \equiv R_n(X). \tag{1.13}$$

Thus two inclusive chains of variable coefficients arise, corresponding to even and odd $n$, respectively. Inversion of the point transformation (1.6) allows the arbitrary constants to be amenable as fitting parameters for given wave speeds $C(x)$ for PDE (1.2), drifts $F(x)$ for PDE (1.3), and diffusivities $K(x)$ for (1.4). In particular from a PDE (1.2) with a constant wave speed one can obtain general solutions for inclusive chains of inhomogeneous wave equations.

In like manner a second sequence $\{\tilde{R}_n(X), \tilde{U}_n(X, T)\}$ can be constructed. It arises by simply interchanging $U_1$ and $U_2$ in Eqs. (1.8). For this sequence $\tilde{R}_n(X) \equiv R_n(X)$ but $\tilde{U}_n(X, T)$ is not simply related to $U_n(X, T)$.

Our results directly lead to inclusive chains for second-order linear ordinary differential equations which are reduced forms of PDEs (1.2)–(1.4).

The rest of the paper is outlined as follows:

In Section 2 we present our procedure for the prototype PDE (1.5). In Sections 3 to 5 we apply our procedure to the inhomogeneous wave equation, the Fokker–Planck equation, and the diffusion equation. Related works are discussed in the final section.

## 2. Sequences for the Canonical PDE (1.5)

We construct the two sequences for the canonical PDE (1.5).

For the first sequence we start with a variable coefficient $R_1(X)$. Let $\phi_1(X) = \phi(X; \alpha_1)$ be the general solution of

$$\frac{d\phi_1}{dX} + [\phi_1(X)]^2 = -R_1(X). \tag{2.1}$$

If $U_1(X, T) = U(X, T)$ satisfies PDE (1.5) with variable coefficient $R_1(X)$, then Eq. (1.8) with $\phi(X; \alpha_1) = \phi_1(X)$ given by Eq. (2.1) defines a Bäcklund transformation to determine $U_2(X, T)$, where $U_2(X, T)$ satisfies

$$\frac{\partial^2 U_2}{\partial x^2} + R_2(X) U_2 = \frac{\partial^\mu U_2}{\partial T^\mu}, \tag{2.2a}$$

$$R_2(X) = \frac{d\phi_1}{dX} - [\phi_1(X)]^2. \tag{2.2b}$$

The sequence $\{R_n(X), U_n(X, T)\}$ proceeds as follows:

$$R_n(X) = \frac{d\phi_{n-1}}{dX} - [\phi_{n-1}(X)]^2, \qquad n = 2, 3, ..., \tag{2.3}$$

where

$$\phi_n(X) = \phi(X; \alpha_1, \alpha_2, ..., \alpha_n) \tag{2.4}$$

is the general solution of the Riccati equation

$$\frac{d\phi_n}{dX} + [\phi_n(X)]^2 = -R_n(X), \qquad n = 1, 2, .... \tag{2.5}$$

Correspondingly $U_n(X, T)$ is determined by the Bäcklund transformation pair:

$$\frac{\partial^{\mu-1} U_n(X, T)}{\partial T^{\mu-1}} = \frac{\partial U_{n-1}(X, T)}{\partial X} - \phi_{n-1}(X) U_{n-1}(X, T), \tag{2.6a}$$

$$\frac{\partial U_{n-1}(X, T)}{\partial T} = \frac{\partial U_n(X, T)}{\partial X} + \phi_{n-1}(X) U_n(X, T), \tag{2.6b}$$

$n = 2, 3, ....$ $U_n(X, T)$ satisfies

$$\frac{\partial^2 U_n}{\partial X^2} + R_n(X) U_n = \frac{\partial^\mu U_n}{\partial T^\mu}, \tag{2.7}$$

$n = 1, 2, 3, ....$

The second sequence $\{\tilde{R}_n(X), \tilde{U}_n(X, T)\}$ is found by exploiting the reciprocal nature of (1.9) and (1.10). We label the given variable coefficient by $\tilde{R}_1(X)$. Then

$$\tilde{R}_n(X) = -[\tilde{\phi}_{n-1}(X)]^2 - \frac{d\tilde{\phi}_{n-1}}{dX}, \qquad n = 2, 3, ..., \tag{2.8}$$

where

$$\tilde{\phi}_n(X) = \tilde{\phi}(X; \alpha_1, \alpha_2, ..., \alpha_n) \tag{2.9}$$

is the general solution of the Riccati equation

$$\frac{d\tilde{\phi}_n}{dX} - [\tilde{\phi}_n(X)]^2 = \tilde{R}_n(X), \qquad n = 1, 2, .... \tag{2.10}$$

Correspondingly

$$\tilde{U}_1(X, T) = U(X, T) \tag{2.11}$$

and $\tilde{U}_n(X, T)$ is the solution of the Bäcklund equation pair

$$\frac{\partial^{\mu-1}\tilde{U}_{n-1}(X, T)}{\partial T^{\mu-1}} = \frac{\partial \tilde{U}_n(X, T)}{\partial X} - \tilde{\phi}_{n-1}(X)\tilde{U}_n(X, T), \tag{2.12a}$$

$$\frac{\partial \tilde{U}_n(X, T)}{\partial T} = \frac{\partial \tilde{U}_{n-1}(X, T)}{\partial X} + \tilde{\phi}_{n-1}(X)\tilde{U}_{n-1}(X, T), \tag{2.12b}$$

$n = 2, 3, \ldots$. $\tilde{U}_n(X, T)$ satisfies

$$\frac{\partial^2 \tilde{U}_n}{\partial X^2} + \tilde{R}_n(X)\tilde{U}_n = \frac{\partial^\mu \tilde{U}_n}{\partial T^\mu}, \tag{2.13}$$

$n = 1, 2, 3, \ldots$.

Before proceeding further note that one can solve explicitly all Riccati equations (2.5), (2.10) for $n = 2, 3, \ldots$. Details follow.

For the first sequence $\{R_n(X), U_n(X, T)\}$, Eqs. (2.5) and (2.3) determine $\{\phi_n(X)\}$ and hence $\{R_n(X)\}$. $\phi_1(X)$ is the general solution of the Riccati equation

$$\frac{d\phi_1}{dX} + [\phi_1(X)]^2 = -R_1(X), \tag{2.14}$$

and $Z = \phi_n(X)$ is the general solution of

$$\frac{dZ}{dX} + Z^2 = [\phi_{n-1}(X)]^2 - \frac{d\phi_{n-1}}{dX}, \tag{2.15}$$

$n = 2, 3, \ldots$. Note that

$$Z = -\phi_{n-1}(X) \tag{2.16}$$

is a particular solution of the Riccati equation (2.15). Hence its general solution $\phi_n(X)$ can be determined explicitly:

$$\phi_n(X) = -\phi_{n-1}(X) + \chi_n(X), \tag{2.17}$$

where

$$\begin{aligned} \chi_n(X) &= \chi(X; \alpha_1, \alpha_2, \ldots, \alpha_n) \\ &= \frac{\Gamma_{n-1}(X)}{[\int_0^X \Gamma_{n-1}(\xi)\, d\xi + 1/\alpha_n]}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \Gamma_{n-1}(X) &= \Gamma(X; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \\ &= \exp\left[2\int_0^X \phi_{n-1}(\xi)\, d\xi\right], \end{aligned} \tag{2.19}$$

and $\alpha_n$ is an arbitrary constant, $n = 2, 3, \ldots$. Note that any scaling of $\Gamma_{n-1}(X)$ corresponds to a scaling of $\alpha_n$ and that at any stage the lower limits of integration in (2.18) can be fixed at numbers of convenience by a translation of $1/\alpha_n$.

Now the sequence $\{R_n(X)\}$ can be determined from (2.5) and (2.17)–(2.19):

$$R_2(X) = -2[\phi_1(X)]^2 - R_1(X), \tag{2.20a}$$

$$R_{n+1}(X) = R_{n-1}(X) - 2\chi_n(X)[\chi_n(X) - 2\phi_{n-1}(X)], \tag{2.20b}$$

$n = 2, 3, \ldots$.

From (2.17) and (2.19) and the observation that (2.18) can be rewritten as

$$\chi_n(X) = \frac{d}{dX}\log\left|\int_0^X \Gamma(\xi)\,d\xi + \frac{1}{\alpha_n}\right|, \tag{2.21}$$

one can show that the two quadratures in (2.18), (2.19) can be reduced to a single quadrature from the derived relation

$$\Gamma_n(X) = \left[\frac{1}{\chi_n(X)}\right]^2 \Gamma_{n-1}(X), \qquad n = 2, 3, \ldots, \tag{2.22}$$

using our freedom to scale $\Gamma_{n-1}(X)$. In particular the sequence $\{\Gamma_n(X), \chi_n(X)\}$ is

$$\Gamma_1(X) = \exp\left[2\int_0^X \phi_1(\xi)\,d\xi\right], \tag{2.23a}$$

$$\chi_2(X) = \frac{d}{dX}\log\left|\int_0^X \Gamma_1(\xi)\,d\xi + \frac{1}{\alpha_2}\right|, \tag{2.23b}$$

$$\Gamma_2(X) = \left[\frac{1}{\chi_2(X)}\right]^2 \Gamma_1(X), \tag{2.23c}$$

$$\chi_3(X) = \frac{d}{dX}\log\left|\int_0^X \Gamma_2(\xi)\,d\xi + \frac{1}{\alpha_3}\right|, \tag{2.23d}$$

etc. The sequence $\{\phi_n(X), R_n(X)\}$ follows immediately from (2.17) and (2.20b), respectively.

Two *inclusive* chains for $\{R_n(X)\}$ naturally arise from the above procedure for odd and even $n$, respectively. In particular it is straightforward to establish the following inclusive chains $\{R_{2n-1}\}$, $\{R_{2n}\}$:

$$R_{2n+1}=R_1-2\sum_{k=1}^{n}\chi_{2k}[\chi_{2k}-2\phi_{2k-1}], \tag{2.24a}$$

$$R_{2n+2}=R_2-2\sum_{k=1}^{n}\chi_{2k+1}[\chi_{2k+1}-2\phi_{2k}], \tag{2.24b}$$

$n=1, 2, \ldots$. These chains are inclusive in the sense that if $\alpha_{n+1}=0$, then $\chi_{n+1}(X)\equiv 0$ and hence

$$R_{n+2}(X)\equiv R_n(X).$$

The solution sequences $\{U_{2n}(X, T)\}$, $\{U_{2n-1}(X, T)\}$ for the corresponding inclusive chains of equations of form (1.5) can be derived from repeated use of Eq. (2.6), and then use of Eqs. (2.7), (2.3), and (2.17).

This completes the discussion of the first sequence $\{R_n(X), U_n(X, T)\}$ arising from Eqs. (2.2)–(2.6).

Finally, in this section we consider the second sequence $\{\tilde{R}_n(X), \tilde{U}_n(X, T)\}$ defined by Eqs. (2.8)–(2.12). Here $\tilde{\phi}_1(X)$ is the general solution of the Riccati equation

$$\frac{d\tilde{\phi}_1}{dX}-[\tilde{\phi}_1]^2=\tilde{R}_1(X) \tag{2.25}$$

and $\tilde{Z}=\tilde{\phi}_n(X)$ is the general solution of

$$\frac{d\tilde{Z}}{dX}-\tilde{Z}^2=-[\tilde{\phi}_{n-1}(X)]^2-\frac{d\tilde{\phi}_{n-1}}{dX}, \tag{2.26}$$

$n=2, 3, \ldots$. To compare the first and second sequences suppose that $\tilde{R}_1(X)\equiv R_1(X)$. Comparison of (2.14) and (2.25) shows that

$$\tilde{\phi}_1\equiv -\phi_1. \tag{2.27}$$

By induction

$$\tilde{\phi}_n\equiv -\phi_n, \qquad n=2, 3, \ldots, \tag{2.28}$$

and from (2.3) and (2.8) it immediately follows that

$$\tilde{R}_n(X)\equiv R_n(X). \tag{2.29}$$

Next we relate the solutions $\{\tilde{U}_n(X, T)\}$ and $\{U_n(X, T)\}$, where $\tilde{U}_1(X, T) \equiv U_1(X, T)$. From (2.6), (2.12), and (2.27) we get

$$\frac{\partial^{\mu-1} U_2}{\partial T^{\mu-1}} = \frac{\partial U_1}{\partial X} - \phi_1 U_1, \tag{2.30a}$$

$$\frac{\partial^{\mu-1} U_1}{\partial T^{\mu-1}} = \frac{\partial \tilde{U}_2}{\partial X} + \phi_1 \tilde{U}_2, \tag{2.30b}$$

$$\frac{\partial \tilde{U}_2}{\partial T} = \frac{\partial U_1}{\partial X} - \phi_1 U_1. \tag{2.30c}$$

When $\mu = 1$

$$U_2 \equiv \frac{\partial \tilde{U}_2}{\partial T}, \tag{2.31}$$

and inductively one can show that

$$U_n \equiv \frac{\partial^{n-1} \tilde{U}_n}{\partial T^{n-1}}, \qquad n = 1, 2, \ldots. \tag{2.32}$$

Thus $U_n$ can be determined trivially from $\tilde{U}_n$ but the converse is not true. An example is given in Bluman and Reid [4].

When $\mu = 2$ one can show that

$$\frac{\partial^{n-1} U_n(X, T)}{\partial T^{n-1}} \equiv \frac{\partial^{n-1} \tilde{U}_n(X, T)}{\partial T^{n-1}}, \qquad n = 1, 2, \ldots, \tag{2.33}$$

and no simple relationhip exists between $U_n$ and $\tilde{U}_n$.

If solutions of Eq. (1.5) are sought in the separated form

$$U(X, T) = Y(X) e^{-\omega T}, \tag{2.34}$$

then $Y(X)$ satisfies the ODE

$$\frac{d^2 Y(X)}{dX^2} + [\lambda + R(X)]\, Y(X) = 0, \tag{2.35}$$

where $\lambda = -(-\omega)^{\mu}$. Unlike the PDE case both solution sequences are identical and the transformations (2.6) and (2.12) are equivalent to the single transformation

$$Y_n(X; \lambda) = \frac{dY_{n-1}(X; \lambda)}{dX} - \phi_{n-1}(X)\, Y_{n-1}(X; \lambda), \tag{2.36}$$

$n = 2, 3, \ldots$.

## 3. Sequences for the Inhomogeneous Wave Equation

To apply our procedure to the wave equation (1.2) we use a point transformation of form (1.6) to transform it to the canonical form (1.5) with $\mu=2$. Sequences $\{R_n(X), U_n(X, T)\}$ established for the canonical form yield corresponding sequences $\{C_n(x), u_n(x, t)\}$ for (1.2) by inversion of the point transformation (1.6). We only consider the first sequence.

The point transformation (1.6) which brings the wave equation to the canonical form (1.5) is

$$X=\int_0^x \frac{d\xi}{C(\xi)}, \tag{3.1a}$$

$$T=t, \tag{3.1b}$$

$$U(X, T)=[C(x)]^{-1/2}u(x, t). \tag{3.1c}$$

Define

$$\hat{C}(X)=C(x). \tag{3.2}$$

Then $U(X, T)$ satisfies

$$\frac{\partial^2 U}{\partial X^2}+R(X)U=\frac{\partial^2 U}{\partial T^2}, \tag{3.3}$$

where

$$R(X)=-\frac{d\tau}{dX}-[\tau(X)]^2, \tag{3.4}$$

$$\tau(X)=\frac{-1}{2\hat{C}(X)}\frac{d\hat{C}(X)}{dX}. \tag{3.5}$$

Suppose that the first wave equation of a sequence has wave speed $C_1(x)=\hat{C}_1(X)$ and that $\tau_1(X)$ is defined by

$$\tau_1(X)=\frac{-1}{2\hat{C}_1(X)}\frac{d\hat{C}_1(X)}{dX}. \tag{3.6}$$

Then the first Riccati equation of the sequence (2.5) for the wave equation is

$$\frac{d\phi_1}{dX}+[\phi_1(X)]^2=-R_1(X)=\frac{d\tau_1}{dX}+[\tau_1(X)]^2, \tag{3.7}$$

which has the particular solution $\phi_1(X; 0) \equiv \tau_1(X)$. Consequently we can solve the first Riccati equation of such a sequence, unlike the case for the canonical equation (1.5).

The transformation from $u_n(x, t)$ to $u_{n+1}(x, t)$ is defined implicitly by Eq. (2.6) with $\mu = 2$:

$$\frac{\partial U_{n+1}(X, T)}{\partial T} = \frac{\partial U_n(X, T)}{\partial X} + \phi_n(X) U_n(X, T), \tag{3.8a}$$

$$\frac{\partial U_n(X, T)}{\partial T} = \frac{\partial U_{n+1}(X, T)}{\partial X} + \phi_n(X) U_{n+1}(X, T), \tag{3.8b}$$

$n = 1, 2, \ldots$. In order to simplify the integration of Eqs. (3.8) note that

$$\bar{U}_n(X, T) = \frac{\partial U_n(X, T)}{\partial T} \tag{3.9}$$

satisfies the same second-order PDE as $U_n(X, T)$. Then

$$U_{n+1}(X, T) = \frac{\partial \bar{U}_n(X, T)}{\partial X} - \phi_n(X) \bar{U}_n(X, T) \tag{3.10}$$

leads to a solution

$$u_{n+1}(x, t) = [C_{n+1}(x)]^{1/2} U_{n+1}(X, T) \tag{3.11}$$

for the wave equation with wave speed $C_{n+1}(x)$, where $X$ is given by

$$X = \int_0^x \frac{d\xi}{C_{n+1}(\xi)}. \tag{3.12}$$

The following formulae are helpful:

$$\phi_1(X) = \tau_1(X) + \frac{d}{dX} \log \left| \int_0^X \hat{C}_1(\eta)\, d\eta + \frac{1}{\alpha_1} \right|, \tag{3.13}$$

$$\Gamma_1(X) = \frac{(\alpha_1)^2}{\hat{C}_1(X)} \left[ \int_0^X \hat{C}_1(\eta)\, d\eta + \frac{1}{\alpha_1} \right]^2, \tag{3.14}$$

$$\hat{C}_{n+1}(X) = \frac{1}{\hat{C}_n(X)} \left[ 1 + \alpha_{n+1} \int_0^X \frac{d\eta}{\hat{C}_n(\eta)} \right]^{-2}, \tag{3.15}$$

$$\phi_n(X) = -\phi_{n-1}(X) + \chi_n(X), \tag{3.16}$$

where $\chi_n(X)$ is given by (2.18) and (2.19), $n = 2, 3, \ldots$. If $\alpha_1 = 0$, then (3.15) also holds for $n = 1$.

By using Eq. (3.15) we obtain

$$x = \int_0^X \frac{d\eta}{\hat{C}_{n-1}(\eta)} \left[ 1 + \alpha_n \int_0^X \frac{d\eta}{\hat{C}_{n-1}(\eta)} \right]^{-1}, \qquad n = 2, 3, ..., \tag{3.17}$$

as the transformation linking $X$ and $x$ for the wave equation with wave speed $C_n(x)$.

If $\alpha_{n+1} = 0$,

$$\hat{C}_{n+1}(X) = \frac{1}{\hat{C}_n(X)}. \tag{3.18}$$

Hence if $\alpha_{n+2} = \alpha_{n+1} = 0$,

$$\hat{C}_{n+2}(X) \equiv C_n(X). \tag{3.19}$$

Thus two inclusive chains result: $\{C_{2n}(x), u_{2n}(x, t)\}$, $\{C_{2n-1}(x), u_{2n-1}(x, t)\}$.

### 3.1. *Examples of Sequences* $\{C_n(x), u_n(x, t)\}$ *for the Wave Equation*

The most important sequence originates from the constant speed wave equation where $C_1(x) = 1$. From its known general solution $u_1(x, t)$ we can explicitly obtain the corresponding general solution $u_n(x, t)$ for the wave equation (1.2) with wave speed $C_n(x)$. For $n \geqslant 2$, $C_n(x)$ can depend on $n$ arbitrary constants.

The wave equation with wave speed $C_1(x) = 1$ has general solution

$$u_1(x, t) = f_1(x - t) + g_1(x + t), \tag{3.20}$$

where $f_1$ and $g_1$ are arbitrary functions of their respective arguments. It follows that

$$U_1(X, T) = f_1(X - T) + g_1(X + T). \tag{3.21}$$

We consider the case $\alpha_1 = 0$. Equation (3.15) implies that

$$\hat{C}_2(X) = [1 + \alpha_2 X]^{-2}, \tag{3.22}$$

and inversion of Eq. (3.17) yields

$$X = \frac{x}{1 - \alpha_2 x}. \tag{3.23}$$

From Eq. (3.10)

$$U_2(X, T) = f_2(X - T) + g_2(X + T), \tag{3.24}$$

where $f_2$ and $g_2$ are arbitrary functions of their respective arguments. Equation (3.11) yields

$$C_2(x) = (1 - \alpha_2 x)^2, \tag{3.25}$$

$$u_2(x, t) = (1 - \alpha_2 x)\left[f_2\left(\frac{x}{1 - \alpha_2 x} - t\right) + g_2\left(\frac{x}{1 - \alpha_2 x} + t\right)\right]. \tag{3.26}$$

This general solution has appeared previously in [5–7].

For $n = 2$, Eq. (3.15) yields

$$\hat{C}_3(X) = (1 + \alpha_2 X)^2\left[1 + \frac{\alpha_3}{3\alpha_2}\{(1 + \alpha_2 X)^3 - 1\}\right]^{-2}, \tag{3.27}$$

and following our scheme we get

$$C_3(x) = (1 - \alpha_3 x)^{4/3}(1 - \beta_3 x)^{2/3}, \tag{3.28}$$

$$u_3(x, t) = (1 - \alpha_3 x)^{2/3}(1 - \beta_3 x)^{1/3}\left[f_3'(X - t) + g_3'(X + t) - \frac{\alpha_2}{1 + \alpha_2 X}\{f_3(X - t) + g_3(X + t)\}\right], \tag{3.29}$$

where

$$X = \frac{1}{\alpha_2}\left(\left[\frac{1 - \beta_3 x}{1 - \alpha_3 x}\right]^{1/3} - 1\right), \tag{3.30a}$$

$$\beta_3 = \alpha_3 - 3\alpha_2. \tag{3.30b}$$

In Eq. (3.29), $f_3$ and $g_3$ are arbitrary functions of their arguments. The fact that a general solution is found for wave speed (3.28) appears to be new. The two subcases $\alpha_3 = 0$ and $\alpha_3 = 3\alpha_2$ appear in [7].

At the next step

$$\hat{C}_4(X) = \frac{1}{\hat{C}_3(X)}\left[1 + \alpha_4 \int_0^X \frac{d\eta}{\hat{C}_3(\eta)}\right]^{-2}, \tag{3.31}$$

where

$$\int_0^X \frac{d\eta}{\hat{C}_3(\eta)} = \frac{1}{9\alpha_2^3}\left[\frac{-\beta_3^2}{z} + \alpha_3\beta_3 z^2 + \frac{\alpha_3^2 z^5}{5}\right]_{z=1}^{z=1+\alpha_2 X}. \tag{3.32}$$

Here $X$, and hence $C_4(x)$, is defined implicitly in terms of $x$ by inversion of the relation

$$x = \int_0^X \frac{d\eta}{\hat{C}_3(\eta)}\left[1 + \alpha_4 \int_0^X \frac{d\eta}{\hat{C}_3(\eta)}\right]^{-1}. \tag{3.33}$$

The general solution of the wave equation with wave speed $C_4(x)$ is obtained by using Eq. (3.10) with $n=3$ and

$$\phi_3(X)=\frac{-\alpha_2}{1+\alpha_2 X}+\frac{3\alpha_2\alpha_3(1+\alpha_2 X)^2}{3\alpha_2-\alpha_3+\alpha_3(1+\alpha_2 X)^3}. \tag{3.34}$$

This process can be continued indefinitely to produce two inclusive chains: $\{C_{2n}(x), u_{2n}(x, t)\}$, $\{C_{2n-1}(x), u_{2n-1}(x, t)\}$.

In general the relationship between $x$ and $X$ for members of these chains will be implicit, and the integrals involved will not be elementary. An exceptional case occurs, however, if we put

$$\alpha_k=(2k-3)\alpha_2, \qquad k=2, 3, ..., n-1, \quad n\geqslant 3. \tag{3.35}$$

Then

$$C_n(x)=(1-\alpha_n x)^{2-(2n-4)/(2n-3)}(1-\beta_n x)^{(2n-4)/(2n-3)}, \tag{3.36}$$

$$u_n(x, t)=[C_n(x)]^{1/2}\left\{\prod_{j=1}^{j=n-2}\left(\frac{\partial}{\partial X}-\frac{j\alpha_2}{1+\alpha_2 X}\right)\right\}(f(X-t)+g(X+t)), \tag{3.37}$$

where

$$X=\frac{1}{\alpha_2}\left(\left[\frac{1-\beta_n x}{1-\alpha_n x}\right]^{1/(2n-3)}-1\right), \tag{3.38a}$$

$$\beta_n=\alpha_n-(2n-3)\alpha_2, \qquad n=3, 4, .... \tag{3.38b}$$

The case $\alpha_2=0$ is obtained by taking appropriate limits in Eqs. (3.36)–(3.38). When either $\alpha_n=0$ or $\beta_n=0$ in (3.36) we obtain the well-known representations of Darboux [7, 8]:

$$C_n(x)=(1-\beta_n x)^{(2n-4)/(2n-3)}, \tag{3.39}$$

and

$$C_n(x)=(1-\alpha_n x)^{2-(2n-4)/(2n-3)}. \tag{3.40}$$

## 4. Sequences for the Fokker–Planck Equation

We derive sequences of drifts $\{F_n(x)\}$ for the Fokker–Planck equation. The transformation

$$X=x, \tag{4.1a}$$

$$T=t, \tag{4.1b}$$

$$U(X, T)=\exp\left\{\frac{1}{2}\int_0^X F(\xi)\,d\xi\right\}u(x, t) \tag{4.1c}$$

maps the Fokker–Planck equation (1.3) into the canonical form (1.5) with $\mu = 1$. Consequently no inverse transformations are involved. Moreover we can solve explicitly the first Riccati equation of any sequence. The results are summarized as follows:

Let $u_1(x, t)$ be a solution of Eq. (1.3) with drift $F(x) = F_1(x)$. Then

$$F_{n+1}(x) = -F_n(x) + \Omega_n(x), \tag{4.2}$$

where

$$\Omega_n(x) = \frac{2\Delta_n(x)}{\int_0^x \Delta_n(\xi)\, d\xi + 1/\alpha_n}, \tag{4.3}$$

$$\Delta_n(x) = \exp\left[\int_0^x F_n(\xi)\, d\xi\right], \tag{4.4}$$

to within the same scale factors mentioned previously, and $\alpha_n$ is an arbitrary constant, $n = 1, 2, \ldots$.

Correspondingly the solution $u_{n+1}(x, t)$ of Eq. (1.3) with drift $F(x) = F_{n+1}(x)$ is

$$u_{n+1}(x, t) = \Delta_n(x) \exp\left[-\frac{1}{2}\int \Omega_n(x)\, dx\right] \cdot \left[\left[F_n(x) - \frac{1}{2}\Omega_n(x)\right] u_n(x, t) + \frac{\partial u_n}{\partial x}\right], \tag{4.5}$$

$n = 1, 2, \ldots$.

Note that $F_{n+2}(x) \equiv F_n(x)$ if $\alpha_{n+1} = \alpha_n = 0$. Hence the sequences $\{F_{2n}(x)\}$ and $\{F_{2n-1}(x)\}$ are inclusive chains of drifts.

One can adapt our scheme to find Green's functions for each element of a sequence from knowledge of the Green's function for the Fokker–Planck equation with drift $F_1(x)$.

## 5. Sequences for the Diffusion Equation

Sequences $\{K_n(x), u_n(x, t)\}$ for the diffusion equation (1.4) are found as follows:

The transformation

$$X = \int_0^x [K(\xi)]^{-1/2}\, d\xi, \tag{5.1a}$$

$$T = t, \tag{5.1b}$$

$$U(X, T) = [K(x)]^{1/4} u(x, t) \tag{5.1c}$$

maps the diffusion equation (1.4) to the canonical form (1.5) with $\mu = 1$.

Let

$$\hat{K}(X) = K(x). \tag{5.2}$$

Then $U(X, T)$ satisfies

$$\frac{\partial^2 U}{\partial X^2} + R(X)U = \frac{\partial U}{\partial T}, \tag{5.3}$$

where

$$R(X) = -\frac{d\sigma}{dX} - [\sigma(X)]^2, \tag{5.4}$$

$$\sigma(X) = \frac{1}{4\hat{K}(X)} \frac{d\hat{K}(X)}{dX}. \tag{5.5}$$

Suppose that the first diffusion equation of a sequence has diffusivity $K_1(x) = \hat{K}_1(X)$. Let

$$\sigma_1(X) = \frac{1}{4\hat{K}_1(X)} \frac{d\hat{K}_1(X)}{dX}. \tag{5.6}$$

Then the first Riccati equation of sequence (2.5) is

$$\frac{d\phi_1}{dX} + [\phi_1(X)]^2 = -R_1(X) = \frac{d\sigma_1}{dX} + [\sigma_1(X)]^2 \tag{5.7}$$

with particular solution $\phi_1(X; 0) \equiv \sigma_1(X)$.

Then

$$U_{n+1}(X, T) = \frac{\partial U_n(X, T)}{\partial X} - \phi_n(X) U_n(X, T), \tag{5.8}$$

$n = 1, 2, \ldots$. We note the following:

$$\phi_1(X) = \sigma_1(X) + \frac{d}{dX} \log \left| \int_0^X [\hat{K}_1(\eta)]^{-1/2} \, d\eta + \frac{1}{\alpha_1} \right|, \tag{5.9}$$

$$\Gamma_1(X) = [\hat{K}_1(X)]^{1/2} \left[ \int_0^X [\hat{K}_1(\eta)]^{-1/2} \, d\eta + \frac{1}{\alpha_1} \right]^2, \tag{5.10}$$

$$\hat{K}_{n+1}(X) = \frac{1}{\hat{K}_n(X)} \left[ 1 + \alpha_{n+1} \int_0^X [\hat{K}_n(\eta)]^{1/2} \right]^4, \tag{5.11}$$

$$\phi_n(X) = -\phi_{n-1}(X) + \chi_n(X), \qquad n = 2, 3, \ldots, \tag{5.12}$$

where $\chi_n(X)$ is given by (2.18), (2.19), and $\hat{K}_n(X) \equiv K_n(x)$. The transformation linking $X$ and $x$ for the diffusion equation with diffusivity $K_n(x)$ is

$$x = \int_0^X [\hat{K}_{n-1}(\eta)]^{1/2} \, d\eta \left[ 1 + \alpha_n \int_0^X [\hat{K}_{n-1}(\eta)]^{1/2} \, d\eta \right]^{-1}, \quad n = 2, 3, \ldots. \tag{5.13}$$

Again two inclusive chains can be obtained.

## 6. Related Works

The idea of forming sequences of solutions for related PDEs is not new. In his celebrated work [1] Bäcklund produced inclusive chains of solutions for the sine-Gordon equation. Each successive number of a chain depended on an additional constant. More recently such auto-Bäcklund transformations have been used to obtain multi-soliton solutions for a variety of nonlinear PDEs (cf. [9, 10] for literature surveys).

Darboux [8] used the theory of Laplace invariants [11] to generate sequences of linear hyperbolic PDEs (Laplace series) by a Bäcklund transformation. However, his method does not generate new constants.

### 6.1. *The Generalized Bäcklund Transformations of Loewner*

The most closely related work to our own appears to be that of Loewner [2, 3]. Loewner considered a system of two first-order PDEs equivalent to the inhomogeneous wave equation (1.2) to within a point transformation of the $x$-variable. His generalization of a Bäcklund transformation (called a Loewner transformation in [9]) maps a pair of such first-order PDEs to another pair of such first-order PDEs. These transformations involve four coupled first-order PDEs. In principle his method generates sequences of inhomogeneous wave equations of the form (1.2) containing new arbitrary constants at each stage.

The objective of his method was to find the most general Loewner transformation preserving the form of PDE (1.2). By using integrability conditions and several simplifying assumptions he implicitly found particular classes of transformations connected to general solutions of Riccati equations. In particular Loewner generated second elements for sequences arising from the constant speed wave equation and Tricomi's equation, respectively.

For the inhomogeneous wave equation (1.2) we can solve explicitly all Riccati equations; Loewner's method also involves Riccati equations but he gives no algorithm for their explicit solution. For PDEs of the form (1.2) we conjecture that with much further work one should obtain our sequences as an explicit subcase of Loewner's implicit procedure. Numerous applications which can be related to the first step of Loewner's procedure are listed in Chapter 3 of [9].

### 6.2. *Works Related to the ODE Form of the Theory*

The ODE form of our theory arising from the separated form of solutions (2.34) has many connections with existing works. Most of these works correspond to specific choices of the constants in the Bäcklund transformation (2.36). Two exceptions are the works of Wadati *et al.* [12] and Mielnik [13]. In both cases these authors, starting from a specific ODE, find a related ODE whose variable coefficient depends on an arbitrary constant.

#### 6.2.1. *Factorization Method*

The operators $d/dX \pm \phi(X)$ of our procedure are analogous to the raising and lowering operators appearing in the factorization method of Infeld and Hull [14]. However, only particular variable coefficient differential equations of the form (2.35) can be treated by their method. In contrast we can start with any variable coefficient ODE of the form (2.35) and generate chains of variable coefficient ODEs whose $n$th variable coefficient depends on $n$ constants.

#### 6.2.2. *The Associated Sturm–Liouville Systems of Crum*

A close relationship exists between our work for ODEs and the associated Sturm–Liouville systems of Crum. Crum [15] explicitly obtained complete orthonormal sets of eigenfunctions and eigenvalues for related Sturm–Liouville systems of the form

$$\frac{d^2Y}{dX^2} + [\lambda - Q_n(X)]\, Y = 0, \qquad 0 < X < 1, \tag{6.1}$$

where for $n = 1$,

$$Y'(0) = \alpha Y(0), \qquad Y'(1) = \beta Y(1), \tag{6.2}$$

$\alpha$ and $\beta$ given constants. For $n > 1$ the eigenfunctions resulting from Crum's procedure satisfy

$$Y(0) = Y(1) = 0. \tag{6.3}$$

Given the spectrum for $n=1$,

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \tag{6.4}$$

Crum's spectrum at step $n$ is

$$\lambda_n < \lambda_{n+1} < \lambda_{n+2} < \cdots < \lambda_{n-1+k} < \cdots \tag{6.5}$$

with correponding eigenfunctions $\{Y_k^{(n)}(X; \lambda_{n-1+k})\}$, $k=1, 2, ...$.

Each step of Crum's procedure can be obtained from our procedure. To begin Crum's procedure (step 1) we express

$$\lambda - Q_1(X) = (\lambda - \lambda_1) + (\lambda_1 - Q_1(X)). \tag{6.6}$$

Setting

$$R_1(X) = \lambda_1 - Q_1(X) \tag{6.7}$$

and choosing the *particular* solution

$$\phi_1(X) = \frac{d}{dX} \log |Y_1^{(1)}(X; \lambda_1)| \tag{6.8}$$

of Eq. (2.14) one obtains by Eq. (2.36) with $n=2$ the second ODE of Crum's sequence. In particular the function

$$Y_k^{(2)}(X; \lambda_{k+1}) = \frac{d}{dX} Y_{k+1}^{(1)} - \phi_1(X)\, Y_{k+1}^{(1)}, \tag{6.9}$$

$k=1, 2, ...$, determines a solution of

$$\frac{d^2 Y_k^{(2)}}{dX^2} + [\lambda_{k+1} - Q_2(X)]\, Y_k^{(2)} = 0, \qquad k=1, 2, ..., \tag{6.10}$$

where $Q_2(X)$ is defined by

$$Q_2(X) - \lambda_1 = -\frac{d\phi_1}{dX} + [\phi_1(X)]^2. \tag{6.11}$$

From (6.9), (6.2), and (6.8) it immediately follows that

$$Y_k^{(2)}(0) = Y_k^{(2)}(1) = 0, \qquad k=1, 2, .... \tag{6.12}$$

Consequently we have constructed the eigenfunctions for Crum's spectrum at step 1. Further steps in Crum's procedure follow similarly. His sequences do not contain arbitrary constants but correspond to a special choice of particular solution of the Riccati equation at each step. In Bluman and Reid [4] we show how our generalization can be used to solve eigenvalue problems not solvable by Crum's procedure.

#### 6.2.3. *The Work of Deift and Supersymmetric Quantum Mechanics*

Deift [16] has shown that a large range of problems of mathematical physics can be regarded as the application of a commutation formula involving bounded operators in a Hilbert space. In particular his results when applied to ODEs correspond to an operator formalization of the work of Crum [15].

Recently there have been a number of papers (see [17, 18] and the references cited therein) concerning an application of supersymmetry to quantum mechanics. To a particular one-dimensional quantum system one is able to associate a supersymmetric partner Hamiltonian. Again the underlying transformation is that discovered long ago by Crum [15] and put in an operator context by Deift.

### 6.3. *Other Works*

More details of work presented in this paper appear in Bluman and Reid [4]. Closely connected work appears in Varley and Seymour [19].

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