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Local and nonlocal symmetries for nonlinear telegraph equation

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In this paper, local and nonlocal symmetry classifications are considered for four equivalent nonlinear telegraph equations. A complete potential symmetry classification of a scalar nonlinear telegraph equation is given through the point symmetry classification of a related potential system. Six new classes of equations are shown to admit potential symmetries. The relationships between local (including contact) and nonlocal (potential) symmetries of these equations are explored. A physical example is considered for possible applications of the obtained potential symmetries. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1841481]

I. INTRODUCTION

In Refs. 1–4, an algorithmic procedure has been developed to find nonlocal symmetries (potential symmetries) of partial differential equations (PDEs) to extend the classes of symmetries admitted by PDEs. Various researchers have found examples of PDEs that admit potential symmetries or extended the procedure to find potential systems.[5]

In recent years, there have been several investigations (Refs. 6–8) to find symmetries for nonlinear telegraph equations of the form

$$u_{tt} = [F(u)u_x]_x + [G(u)]_x. \tag{1}$$

PDE (1) is equivalent to the potential system

$$v_x = u_t,$$

$$v_t = F(u)u_x + G(u). \tag{2}$$

In particular, if $(u,v)=(U(x,t),V(x,t))$ solves (2), then $u=U(x,t)$ solves (1). Conversely, for any $u=U(x,t)$ solving (1), there exists a pair of functions $(u,v)=(U(x,t),V(x,t))$ solving (2) with $V(x,t)$ unique to within a constant.

Similarly, the potential system (2) is equivalent to the potential system

$$w_t = v,$$

$$w_x = u, \tag{3}$$

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$$v_t = F(u)u_x + G(u),$$

and hence to the potential equation

$$w_{tt} = F(w_x)w_{xx} + G(w_x). \tag{4}$$

In particular, if $u=U(x,t)$ solves (1), then from the integrability conditions associated with (3) there exists a triplet $(u,v,w)=(U(x,t),V(x,t),W(x,t))$ solving system (3) with $w=W(x,t)$ solving (4). Conversely, if $w=W(x,t)$ solves (4), then $(u,v)=(W_x(x,t),W_t(x,t))$ solves (2) and $u=W_x(x,t)$ solves (1).

Consequently, a symmetry of any one of the PDE systems (1)–(4) defines a symmetry of the remaining three equivalent systems. Moreover, due to the relationship connecting these four equivalent systems, it is possible for a local symmetry of any one of these systems to yield a nonlocal symmetry of another one.[9]

Equations related to (1) include the nonlinear heat conduction equation when $F(u)=0$, and the nonlinear inhomogeneous vibrating string equation when $G(u)=0$.

The group properties of the nonlinear heat equation for both the scalar form (1) and the potential system (2) are presented in Ref. 2. The point symmetry classification of the nonlinear wave equation (1) with $G(u)=0$ and $F(u)$ replaced by $F(x,u)$ is discussed in Ref. 6. The complete point symmetry classifications of equation (1) and the equivalent potential equation (4) are given in Refs. 7 and 8, respectively. Some exact solutions of system (2) are given in Ref. 10 for special forms of $F(u)$ and $G(u)$.

Among the equivalent systems (1)–(4), it appears that the potential system (2) arises most directly in physical situations. One physical example directly related to system (2) is represented by the equations of telegraphy of a two-conductor transmission line with v as the current in the conductors, u as the voltage between the conductors, $G(u)$ as the leakage current per unit length, $F(u)$ as the differential capacitance, t as a spatial variable and x as time.[11] Another physical example related to system (2) is the equation of motion of a hyperelastic homogeneous rod whose cross-sectional area varies exponentially along the rod. Here u is the displacement gradient related to the difference between a spatial Eulerian coordinate and a Lagrangian coordinate x, v is the velocity of a particle displaced by this difference, $G(u)$ is essentially the stress tensor, $F(u)=\lambda G'(u)$ for some constant λ, and t is time (see Refs. 12 and 13).

In this paper we give the complete point symmetry classification of the potential system (2) and compare our results with the complete point symmetry classification of the scalar equation (1) included in Ref. 7 and the complete point symmetry classification of the potential equation (4) given in Ref. 8. In particular, we will show the following.

(I) The point symmetry classifications of the scalar equations (1) and (4) are identical, i.e., for any $F(u)$ and $G(u)$, a point symmetry admitted by (1) induces a point symmetry admitted by (4) and vice versa.

(II) For wide classes of $F(u)$ and $G(u)$, there exist point symmetries of the potential system (2) which are nonlocal symmetries of the scalar equation (1).

(III) Each point symmetry of the potential system (2) which is a nonlocal symmetry of (1) yields a contact symmetry of the potential equation (4) that is *not* a point symmetry of (4).

(IV) For all but one particular class of $F(u)$ and $G(u)$, a point symmetry of the scalar equation (1) is a point symmetry of (2).

In Sec. II, we give the set of determining equations for point symmetries of the potential system (2) and the complete potential symmetry classification of the scalar equation (1) related to (2). In Secs. III and IV, we present the complete point symmetry classifications of the scalar equations (1) and (4) given in Refs. 7 and 8, respectively. In Sec. V, we compare the point symmetry classifications of the systems (1), (2), and (4) by proving theorems that yield statements (I)–(IV). The second physical example is considered in Sec. VI. Further comments are given in Sec. VII.

II. POTENTIAL SYMMETRY CLASSIFICATION OF THE SCALAR EQUATION (1)

Consider the potential system (2).

The point symmetry

$$x^* = x + \varepsilon \xi(x,t,u,v) + O(\varepsilon^2),$$

$$t^* = t + \varepsilon \tau(x,t,u,v) + O(\varepsilon^2),$$

$$u^* = u + \varepsilon \eta(x,t,u,v) + O(\varepsilon^2),$$

$$v^* = v + \varepsilon \phi(x,t,u,v) + O(\varepsilon^2), \tag{5}$$

is admitted by system (2) if and only if it satisfies the determining equations,

$$X^{(1)}(v_x - u_t) = 0,$$

$$X^{(1)}(v_t - F(u)u_x - G(u)) = 0, \tag{6}$$

for *any* (u,v) that solves system (2);

$$X = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v}$$

is the infinitesimal generator of the point symmetry (5);

$$X^{(1)} = X + \eta^{(1)}\frac{\partial}{\partial u_x} + \eta^{(2)}\frac{\partial}{\partial u_t} + \phi^{(1)}\frac{\partial}{\partial v_x} + \phi^{(2)}\frac{\partial}{\partial v_t},$$

with

$$\eta^{(1)} = \frac{D\eta}{Dx} - \frac{D\xi}{Dx}u_x - \frac{D\tau}{Dx}u_t, \quad \eta^{(2)} = \frac{D\eta}{Dt} - \frac{D\xi}{Dt}u_x - \frac{D\tau}{Dt}u_t,$$

$$\phi^{(1)} = \frac{D\phi}{Dx} - \frac{D\xi}{Dx}v_x - \frac{D\tau}{Dx}v_t, \quad \phi^{(2)} = \frac{D\phi}{Dt} - \frac{D\xi}{Dt}v_x - \frac{D\tau}{Dt}v_t,$$

is the first extension (prolongation) of X;

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xx}\frac{\partial}{\partial v_x} + v_{xt}\frac{\partial}{\partial v_t},$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_x} + v_{tt}\frac{\partial}{\partial v_t}$$

are total derivative operators. Note that $\eta^{(i)}, \phi^{(i)}$ are functions of x,t,u,v,u_x,u_t,v_x,v_t, $i=1,2$.

The global one-parameter (ε) Lie group of point symmetries associated with (5) is obtained by solving the initial value problem for the first order system of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}x^*}{\mathrm{d}\varepsilon} = \xi(x^*,t^*,u^*,v^*),$$

$$\frac{\mathrm{d}t^*}{\mathrm{d}\varepsilon} = \tau(x^*,t^*,u^*,v^*),$$

$$\frac{\mathrm{d}u^*}{\mathrm{d}\varepsilon} = \eta(x^*,t^*,u^*,v^*),$$

$$\frac{\mathrm{d}v^*}{\mathrm{d}\varepsilon} = \phi(x^*,t^*,u^*,v^*), \tag{7}$$

with $x^*=x$, $t^*=t$, $u^*=u$, $v^*=v$ at $\varepsilon=0$.

A point symmetry (5) yields a *nonlocal symmetry* of the scalar equation (1) if and only if $\xi_v^2+\tau_v^2+\eta_v^2\neq 0$, i.e., if and only if the infinitesimals ξ, τ, and η have an essential dependence on the potential variable v. Such a nonlocal symmetry is called a *potential symmetry* of scalar equation (1) related to the potential system (2) (for details, see Ref. 3).

The determining equations (6) simplify to

$$\xi_v - \tau_u = 0,$$

$$\eta_u - \phi_v + \xi_x - \tau_t = 0,$$

$$G(u)\eta_v + \eta_t - \phi_x + G(u)\tau_x = 0,$$

$$\xi_u - F(u)\tau_v = 0,$$

$$\phi_u - G(u)\tau_u - F(u)\eta_v = 0,$$

$$G(u)\xi_v + \xi_t - F(u)\tau_x = 0,$$

$$[\phi_v - \tau_t - 2G(u)\tau_v - \eta_u + \xi_x]F(u) - F'(u)\eta = 0,$$

$$[\phi_v - \tau_t - G(u)\tau_v]G(u) - F(u)\eta_x - G'(u)\eta + \phi_t = 0. \tag{8}$$

We now consider the classification problem of finding all $F(u)$, $G(u)$ such that the potential system (2) yields a potential symmetry of (1).

If

$$F(u) = \frac{c}{(au+b)^2}, \quad G(u) = \frac{d}{au+b} + f \tag{9}$$

for arbitrary constants a,b,c,d,f, or if

$$F(u) \text{ is arbitrary}, \quad G(u) = \text{const},$$

then through the potential system (2), the scalar equation admits an infinite number of potential symmetries and the potential system (2) is linearizable by a point transformation (see Refs. 3 and 4).

Various symbolic manipulation algorithms exist to solve the set of determining equations (8) (for example, see Refs. 14 and 15). Using the symmetry manipulation algorithm presented in Ref. 15, one can prove the following results.

Theorem 1: *The scalar equation (1) admits a potential symmetry related to potential system (2) if and only if the functions $F(u)$ and $G(u)$, with $G'(u)\neq 0$, satisfy the system of ODEs,*

$$(c_4u + c_5)F'(u) - 2(c_1 - c_3 - c_2G(u))F(u) = 0, \tag{10}$$

$$(c_4u + c_5)G'(u) + c_2G^2(u) - (c_1 - 2c_3 + c_4)G(u) - c_6 = 0, \tag{11}$$

for any fixed constants $c_1, c_2, \ldots, c_6$ *with* $c_2 \neq 0$.

In the linearizable case (9),

$$c_1 = 0, \quad c_6 = \frac{c_3(c_4 - c_3)}{c_2}.$$

Theorem 2: *For any* $F(u)$, $G(u)$ *satisfying the system of ODEs (10) and (11) with* $c_2 \neq 0$, *the potential system (2) admits the point symmetry (5) with*

$$\xi = c_1x + c_2 \int F(u)\mathrm{d}u,$$

$$\tau = c_3t + c_2v,$$

$$\eta = c_4u + c_5,$$

$$\phi = c_6t + (c_1 - c_3 + c_4)v, \tag{12}$$

and hence the scalar equation (1) admits the corresponding potential symmetry.

Now we find the functions $F(u)$ and $G(u)$ satisfying (10), (11) and the corresponding potential symmetries (12).

Note that the point transformation

$$\overline{x} = ax + b, \quad \overline{t} = ct + d, \quad \overline{u} = \alpha u + \beta, \quad \overline{v} = \gamma v + \rho t$$

for any constants a, b, c, d, α, β, γ, and ρ such that $ac\alpha\gamma \neq 0$ and $a\alpha = c\gamma$ is an equivalence transformation for system (2). Under this transformation, system (2) becomes the equivalent system

$$\overline{v}_{\overline{x}} = \overline{u}_{\overline{t}}, \quad \overline{v}_{\overline{t}} = \overline{F}(\overline{u})\overline{u}_{\overline{x}} + \overline{G}(\overline{u}),$$

where

$$\overline{F}(\overline{u}) = \frac{a\gamma}{c\alpha}F\left(\frac{\overline{u} - \beta}{\alpha}\right)$$

and

$$\overline{G}(\overline{u}) = \frac{\gamma}{c}G\left(\frac{\overline{u} - \beta}{\alpha}\right) + \frac{\rho}{c}.$$

We use such equivalence transformations to simplify the analysis. For example, if $G(u) = a_0(b_0u + c_0)^{d_0} + f_0$, without loss of generality we can assume that $G(u) = u^{d_0}$.

Modulo translations and scalings in u and G, we obtain six distinct classes of ODEs for $F(u)$ and $G(u)$ where scalar equation (1) admits potential symmetries. These six classes of ODEs and their solutions [modulo equivalence classes of $F(u)$ and $G(u)$] are presented in Table I. In Table II for each class we display the corresponding infinitesimals (ξ, τ, η, ϕ) and global group (x^*, t^*, u^*, v^*) obtained from solving the corresponding ODEs (7).

All symmetries presented in Tables I and II are new for each of the equivalent systems (1)–(4). Note that classes 1 and 6 are linearizable[4] if $\beta = 0$ and $\alpha = 1/2$.

TABLE I. Classes of $F(u)$ and $G(u)$ yielding potential symmetries of (1).

Class	ODEs satisfied by $F(u)$ and $G(u)$	$G(u)$	$F(u)$	Relationship between $F(u)$ and $G(u)$
1	$uG'-\alpha(1-G^2)=0$ $uF'-(\beta-1-2\alpha G)F=0$	$(u^{2\alpha}-1)/(u^{2\alpha}+1)$ $(u^{2\alpha}+1)/(u^{2\alpha}-1)$	$4u^{2\alpha+\beta-1}/(u^{2\alpha}+1)^2$ $-4u^{2\alpha+\beta-1}/(u^{2\alpha}-1)^2$	$F(u)=(u^{\beta}/\alpha)G'(u)$
2	$uG'-\alpha(1+G^2)=0$ $uF'+(1-\beta-2\alpha G)F=0$	$\tan(\alpha \ln u)$	$u^{\beta-1}\sec^2(\alpha \ln u)$	”
3	$uG'+G^2=0$ $uF'-(\beta-1-2G)F=0$	$(\ln u)^{-1}$	$-u^{\beta-1}(\ln u)^{-2}$	”
4	$G'-G^2-1=0$ $F'-2(\beta+G)F=0$	$\tan u$	$e^{2\beta u}\sec^2 u$	$F(u)=e^{2\beta u}G'(u)$
5	$G'+G^2-1=0$ $F'-2(\beta-G)F=0$	$\tanh u$ $\coth u$	$e^{2\beta u}\,\mathrm{sech}^2 u$ $-e^{2\beta u}\,\mathrm{csch}^2 u$	”
6	$G'+G^2=0$ $F'-2(\beta-G)F=0$	u^{-1}	$-u^{-2}e^{2\beta u}$	”

III. POINT SYMMETRY CLASSIFICATION OF THE SCALAR EQUATION (1)

In Ref. 7, Kingston and Sophocleous considered the classification problem of finding all $F(u)$, $G(u)$ such that the scalar equation (1) admits a point symmetry. The point symmetry

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),$$

TABLE II. Potential symmetries of (1) for each class $[\Gamma(\beta,F,\varepsilon)=\int^{ue^{2\varepsilon}} s^{-(1+\beta)}(\int^s F(x)\mathrm{d}x)\mathrm{d}s,\ \ \Omega(\beta,F,\varepsilon)=\int^{u+\varepsilon} e^{-2\beta s}(\int^s F(x)\mathrm{d}x)\mathrm{d}s]$.

	Class	Infinitesimals ξ,τ,η,ϕ	Global group
1	$F(u)=4u^{2\alpha+\beta-1}/(u^{2\alpha}+1)^2$ $G(u)=(u^{2\alpha}-1)/(u^{2\alpha}+1)$ or $F(u)=-4u^{2\alpha+\beta-1}/(u^{2\alpha}-1)^2$ $G(u)=(u^{2\alpha}+1)/(u^{2\alpha}-1)$	$\xi=2(\beta x+\alpha\int F(u)\mathrm{d}u)$ $\tau=(\beta+1)t+2\alpha v$ $\eta=2u$ $\phi=2\alpha t+(\beta+1)v$	$x^*=e^{2\beta\varepsilon}[x+\alpha u^{\beta}(\Gamma(\beta,F,\varepsilon)-\Gamma(\beta,F,0))]$ $t^*=(1/2)e^{(\beta+1)\varepsilon}[(t+v)e^{2\alpha\varepsilon}+(t-v)e^{-2\alpha\varepsilon}]$ $u^*=ue^{2\varepsilon}$ $v^*=(1/2)e^{(\beta+1)\varepsilon}[(t+v)e^{2\alpha\varepsilon}-(t-v)e^{-2\alpha\varepsilon}]$
2	$F(u)=u^{\beta-1}\sec^2(\alpha\ln u)$ $G(u)=\tan(\alpha\ln u)$	$\xi=2(\beta x-\alpha\int F(u)\mathrm{d}u)$ $\tau=(\beta+1)t-2\alpha v$ $\eta=2u$ $\phi=2\alpha t+(\beta+1)v$	$x^*=e^{2\beta\varepsilon}[x-\alpha u^{\beta}(\Gamma(\beta,F,\varepsilon)-\Gamma(\beta,F,0))]$ $t^*=e^{(\beta+1)\varepsilon}[t\cos 2\alpha\varepsilon-v\sin 2\alpha\varepsilon]$ $u^*=ue^{2\varepsilon}$ $v^*=e^{(\beta+1)\varepsilon}[v\cos 2\alpha\varepsilon+t\sin 2\alpha\varepsilon]$
3	$F(u)=-u^{\beta-1}(\ln u)^{-2}$ $G(u)=(\ln u)^{-1}$	$\xi=2(\beta x+\int F(u)\mathrm{d}u)$ $\tau=(\beta+1)t+2v$ $\eta=2u$ $\phi=(\beta+1)v$	$x^*=e^{2\beta\varepsilon}[x+u^{\beta}(\Gamma(\beta,F,\varepsilon)-\Gamma(\beta,F,0))]$ $t^*=e^{(\beta+1)\varepsilon}(2v\varepsilon+t)$ $u^*=ue^{2\varepsilon}$ $v^*=ve^{(\beta+1)\varepsilon}$
4	$F(u)=e^{2\beta u}\sec^2 u$ $G(u)=\tan u$	$\xi=2\beta x-\int F(u)\mathrm{d}u$ $\tau=\beta t-v$ $\eta=1$ $\phi=t+\beta v$	$x^*=e^{2\beta\varepsilon}[x-e^{2\beta u}(\Omega(\beta,F,\varepsilon)-\Omega(\beta,F,0))]$ $t^*=e^{\beta\varepsilon}(t\cos\varepsilon-v\sin\varepsilon)$ $u^*=u+\varepsilon$ $v^*=e^{\beta\varepsilon}(v\cos\varepsilon+t\sin\varepsilon)$
5	$F(u)=e^{2\beta u}\,\mathrm{sech}^2 u$ $G(u)=\tanh u$ or $F(u)=-e^{2\beta u}\,\mathrm{csch}^2 u$ $G(u)=\coth u$	$\xi=2\beta x+\int F(u)\mathrm{d}u$ $\tau=\beta t+v$ $\eta=1$ $\phi=t+\beta v$	$x^*=e^{2\beta\varepsilon}[x+e^{2\beta u}(\Omega(\beta,F,\varepsilon)-\Omega(\beta,F,0))]$ $t^*=(1/2)e^{\beta\varepsilon}((t+v)e^{\varepsilon}+(t-v)e^{-\varepsilon})$ $u^*=u+\varepsilon$ $v^*=(1/2)e^{\beta\varepsilon}((t+v)e^{\varepsilon}-(t-v)e^{-\varepsilon})$
6	$F(u)=-u^{-2}e^{2\beta u}$ $G(u)=u^{-1}$	$\xi=2\beta x+\int F(u)\mathrm{d}u$ $\tau=\beta t+v$ $\eta=1$ $\phi=\beta v$	$x^*=e^{2\beta\varepsilon}[x+e^{2\beta u}(\Omega(\beta,F,\varepsilon)-\Omega(\beta,F,0))]$ $t^*=e^{\beta\varepsilon}(t+v\varepsilon)$ $u^*=u+\varepsilon$ $v^*=ve^{\beta\varepsilon}$

TABLE III. Classes of $F(u)$ and $G(u)$ yielding point symmetries of (1).

Class	$G(u)$	$F(u)$	Infinitesimals ξ,τ,η
A	e^u	$e^{(\alpha+1)u}$	$\xi=2\alpha x,\tau=(\alpha-1)t,\eta=2$
B	$u^{\alpha+\beta+1}$	u^α	$\xi=2\beta x,\tau=(\alpha+2\beta)t,\eta=-2u$
C	u^{-1}	u^{-2}	Those in class B and $\xi=e^x,\tau=0,\eta=-ue^x$
D	$\ln u$	u^α	$\xi=2(\alpha+1)x,\tau=(\alpha+2)t,\eta=2u$
E	u	$e^{\alpha u}$	$\xi=2\alpha x,\tau=\alpha t,\eta=2$
F	u^{-3}	u^{-4}	Those in class B and $\xi=0,\tau=t^2,\eta=ut$

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2),$$

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2), \tag{13}$$

is admitted by the scalar equation (1) if and only if it satisfies the determining equation

$$X^{(2)}(u_{tt} - (F(u)u_x)_x - G(u)_x) = 0 \tag{14}$$

for any u solving the scalar equation (1);

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u},$$

is the infinitesimal generator of the point symmetry (13); $X^{(2)}$ is the second extension of X.

From (14), the determining equations for $\xi(x,t,u)$, $\tau(x,t,u)$, and $\eta(x,t,u)$ are given by

$$\xi_u = \tau_x = \tau_u = \eta_{uu} = 0,$$

$$\xi_t - F(u)\tau_x = 0,$$

$$2F(u)(\xi_x - \tau_t) - F'(u)\eta = 0,$$

$$\eta_{tt} - F(u)\eta_{xx} - G'(u)\eta_x = 0,$$

$$2\eta_{tu} - \tau_{tt} + F(u)\tau_{xx} + G'(u)\tau_x = 0,$$

$$2F(u)\eta_{xu} + \xi_{tt} - F(u)\xi_{xx} + 2F'(u)\eta_x - G'(u)(\xi_x - 2\tau_t) + G''(u)\eta = 0.$$

For arbitrary $F(u)$ and $G(u)$, clearly each of the equivalent systems (1)–(4) admits translations in $x(\xi=1,\tau=0)$ and $t(\xi=0,\tau=1)$. For specific $F(u)$ and $G(u)$ with $G'(u)\neq 0$, the results are summarized in Table III.

Note that the classes presented in Ref. 7 where $F(u)=1$, $G(u)=e^{\lambda u}$ as well as $F(u)=1$, $G(u)=u^{\lambda+1}$ can be obtained, respectively, by appropriately scaling u in class A and setting $\alpha=0$ in class B.

IV. POINT SYMMETRY CLASSIFICATION OF THE SCALAR EQUATION (4)

In Ref. 8, the authors considered the classification problem of finding all $F(u)$, $G(u)$ such that the scalar equation (4) admits a point symmetry. The point symmetry

$$x^* = x + \varepsilon\xi(x,t,w) + O(\varepsilon^2),$$

TABLE IV. Classes of $F(u)$ and $G(u)$ yielding point symmetries of (4).

Class	$G(u)=G(w_x)$	$F(u)=F(w_x)$	Infinitesimals ξ,τ,η
A	e^u	$e^{(\alpha+1)u}$	$\xi=2\alpha x,\tau=(\alpha-1)t,\Omega=2(\alpha w+x)$
B	$u^{\alpha+\beta+1}$	u^α	$\xi=2\beta x,\tau=(\alpha+2\beta)t,\Omega=2(\beta-1)w$
C	u^{-1}	u^{-2}	Those in class B and $\xi=e^x,\tau=0,\Omega=0$
D	$\ln u$	u^α	$\xi=2(\alpha+1)x,\tau=(\alpha+2)t,\Omega=2(\alpha+2)w+t^2$
E	u	$e^{\alpha u}$	$\xi=2\alpha x,\tau=\alpha t,\Omega=2\alpha w+t^2+2x$
F	u^{-3}	u^{-4}	Those in class B and $\xi=0,\tau=t^2,\Omega=tw$

$$t^* = t+\varepsilon\tau(x,t,w)+O(\varepsilon^2),$$

$$w^* = w+\varepsilon\Omega(x,t,w)+O(\varepsilon^2), \tag{15}$$

is admitted by the scalar equation (4) if and only if it satisfies the determining equation

$$X^{(2)}(w_{tt}-F(w_x)w_{xx}-G(w_x))=0$$

for any w solving the scalar equation (4);

$$X=\xi(x,t,w)\frac{\partial}{\partial x}+\tau(x,t,w)\frac{\partial}{\partial t}+\Omega(x,t,w)\frac{\partial}{\partial w},$$

is the infinitesimal generator of the point symmetry (15); $X^{(2)}$ is the second extension of X.

Modulo equivalence transformations, the results presented in Ref. 8 are summarized in Table IV.

V. DISCUSSION OF THE SYMMETRY CLASSIFICATIONS

In this section we prove the statements (I)–(IV) presented in the Introduction through proofs of the following four theorems.

Theorem 3: *The point symmetry classifications of the scalar equations (1) and (4) are identical, i.e., for any $F(u)$ and $G(u)$ a point symmetry admitted by (1) induces a point symmetry admitted by (4) and vice versa.*

Proof: The infinitesimals of a point symmetry of the scalar equation (1) are of the form

$$\xi=\xi(x),\quad \tau=\tau(t),\quad \eta=(f(t)-\xi'(x))u+b$$

for some functions $\xi(x)$, $\tau(t)$, $f(t)$ and constant b. A corresponding point symmetry of the potential scalar equation (4) (with $u=w_x$) must have $\xi=\xi(x)$, $\tau=\tau(t)$, $\Omega=\Omega(x,t,w)$ with $\Omega(x,t,w)$ satisfying

$$\Omega^{x(1)}=\eta$$

in terms of its first extended infinitesimal $\Omega^{x(1)}$, i.e.,

$$\Omega^{x(1)}=\frac{D\Omega}{Dx}-\frac{D\xi}{Dx}w_x-\frac{D\tau}{Dx}w_t=\Omega_x+\Omega_w w_x-\xi'(x)w_x=(f(t)-\xi'(x))w_x+b. \tag{16}$$

Hence from Eq. (16), we must have

$$\Omega_x=b,\quad \Omega_w=f(t).$$

Then it is necessary that

$$\Omega = bx + f(t)w + g(t), \tag{17}$$

for some $g(t)$ in order that a point symmetry of the scalar equation (1) yields a point symmetry of the scalar equation (4). From Tables III and IV, it is easy to check that condition (17) is satisfied for each point symmetry of the scalar equation (4).

Conversely, the infinitesimals of a point symmetry of (4) are of the form

$$\xi = \xi(x), \quad \tau = \tau(t), \quad \Omega = f(t)w + \beta(x,t)$$

for some functions $\xi(x)$, $\tau(t)$, $f(t)$, and $\beta(x,t)$. A corresponding point symmetry of the scalar equation (1) with $u=w_x$ must have

$$\xi = \xi(x), \quad \tau = \tau(t), \quad \eta = \eta(x,t,u).$$

In terms of the first extended infinitesimal $\Omega^{x(1)}$, the infinitesimal $\eta(x,t,u)$ must satisfy

$$\eta = \Omega^{x(1)} = \frac{D\Omega}{Dx} - \frac{D\xi}{Dx}w_x - \frac{D\tau}{Dx}w_t = \Omega_x + \Omega_w w_x - \xi'(x)w_x = \beta_x + (f(t) - \xi'(x))w_x. \tag{18}$$

From Tables III and IV, it is easy to check that condition (18) is satisfied for each point symmetry of (1). ■

Theorem 4: *For each of the six classes of $F(u)$ and $G(u)$ listed in Table I, there exist point symmetries of the potential system (2) which are nonlocal symmetries of the scalar equation (1).*

Proof: Since each point symmetry in Table II has $\tau_v \neq 0$, it follows that for all classes of $F(u)$ and $G(u)$ listed in Table I (also repeated in Table II) there exist point symmetries of the potential system (2) which are nonlocal (potential) symmetries of the scalar equation (1). ■

Note that only the class $F(u)=u^{-2}$, $G(u)=u^{-1}$ is common for Tables I and III. This class is linearizable since it admits potential symmetries leading to the linearization of (2) by a point transformation.[4]

Theorem 5: *Each point symmetry of the potential system (2) which is a nonlocal symmetry of the scalar equation (1) yields a contact symmetry of the potential equation (4) that is* not *a point symmetry of (4).*

Proof: A *contact symmetry* of a PDE with dependent variable w and independent variables x and t is defined by

$$x^* = x + \varepsilon\xi(x,t,w,w_x,w_t) + O(\varepsilon^2),$$

$$t^* = t + \varepsilon\tau(x,t,w,w_x,w_t) + O(\varepsilon^2),$$

$$w^* = w + \varepsilon\Omega(x,t,w,w_x,w_t) + O(\varepsilon^2), \tag{19}$$

if and only if

$$\frac{\partial\Omega}{\partial w_x} = \frac{\partial\xi}{\partial w_x}w_x + \frac{\partial\tau}{\partial w_x}w_t, \quad \frac{\partial\Omega}{\partial w_t} = \frac{\partial\xi}{\partial w_t}w_x + \frac{\partial\tau}{\partial w_t}w_t. \tag{20}$$

Let characteristic function $W=\Omega-\xi w_x-\tau w_t$. A contact symmetry (19) is a point symmetry if and only if

$$\frac{\partial^2 W}{\partial(w_x)^2} = \frac{\partial^2 W}{\partial(w_t)^2} = \frac{\partial^2 W}{\partial w_x \partial w_t} = 0. \tag{21}$$

For details, see Chap. 5 of Ref. 3.

A point symmetry of potential system (2) which is a nonlocal symmetry of the scalar equation (1) (see Table II) is of the form

$$\xi = ax + b\int^{u} F(s)\mathrm{d}s, \quad \tau = ct + dv,$$

$$\eta = fu + g, \quad \phi = ht + kv, \tag{22}$$

for some constants a, b, c, d, f, g, h, and k with $d \neq 0$, $a+f=c+k$.

Solving (20), after using the substitution (22), we find that

$$\Omega(x,t,w,w_x,w_t) = b\int^{w_x} sF(s)\mathrm{d}s + \frac{d}{2}w_t^2 + A(x,t,w) \tag{23}$$

yields a contact symmetry for an arbitrary function $A(x,t,w)$.

Now we find $A(x,t,w)$ so that (23) yields a contact symmetry of the potential equation (4). Since $u=w_x$, $v=w_t$, it follows that we must have

$$\eta = \Omega^{x(1)} = \frac{D\Omega}{Dx} - \frac{D\xi}{Dx}w_x - \frac{D\tau}{Dx}w_t,$$

$$\phi = \Omega^{t(1)} = \frac{D\Omega}{Dt} - \frac{D\xi}{Dt}w_x - \frac{D\tau}{Dt}w_t, \tag{24}$$

in terms of first extended infinitesimals $\Omega^{x(1)}$ and $\Omega^{t(1)}$. After solving the equations (24), we find that $A(x,t,w)=gx+(a+f)w+\frac{1}{2}ht^2$ and hence

$$\xi = ax + b\int^{w_x} F(s)\mathrm{d}s, \quad \tau = ct + dw_t,$$

$$\Omega = b\int^{w_x} sF(s)\mathrm{d}s + \frac{d}{2}w_t^2 + gx + (a+f)w + \frac{1}{2}ht^2, \tag{25}$$

defines a contact symmetry of the scalar potential equation (4). Using condition (21), it is clear that (25) is *not* a point symmetry of (4) since $d \neq 0$. ■

Theorem 6: *A point symmetry of the scalar equation (1) yields a point symmetry of the potential equation (2) if and only if the infinitesimals for $F(u)$ and $G(u)$ belong to classes A–E in Table III.*

Proof: A point symmetry $(\xi(x,t,u), \tau(x,t,u), \eta(x,t,u))$ of the scalar equation (1) yields a point symmetry of (2) if and only if the set of determining equations (8) has a solution for $\phi(x,t,u,v)$. A solution of (8) exists if and only if the six integrability conditions involving the second order mixed partial derivatives of $\phi(x,t,u,v)$ are satisfied. Consequently, it is easy to show that a point symmetry of (1) yields a point symmetry of (2) if and only if it satisfies the additional conditions

$$\tau_{tt} = 0, \quad \eta_{xu} + \xi_{xx} = 0, \quad \eta_{tt} = 0, \quad F'(u)\eta_x - 2F(u)\xi_{xx} = 0. \tag{26}$$

The infinitesimals for classes A–E in Table III satisfy (26) but those for class F do not since here $\tau_{tt} \neq 0$. ■

As a consequence of Theorem 6, we see that the point symmetry $X=t^2(\partial/\partial t)+ut(\partial/\partial u)$ for $G(u)=u^{-3}$, $F(u)=u^{-4}$ yields a *nonlocal symmetry* of the potential system (2).

TABLE V. Physical examples $[F(u)=G'(u)]$ yielding potential symmetries of (1).

	Class	Infinitesimals ξ,τ,η,ϕ	Global group
1.1	$G(u)=(u^{2\alpha}-1)/(u^{2\alpha}+1)$	$\xi=2\alpha((u^{2\alpha}-1)/(u^{2\alpha}+1))$	$x^*=x+\ln((u^{2\alpha}e^{4\alpha\varepsilon}+1)/(u^{2\alpha}+1))-2\alpha\varepsilon$
		$\tau=t+2\alpha v$	$t^*=\frac{1}{2}e^{\varepsilon}[(t+v)e^{2\alpha\varepsilon}+(t-v)e^{-2\alpha\varepsilon}]$
		$\eta=2u$	$u^*=ue^{2\varepsilon}$
		$\phi=2\alpha t+v$	$v^*=\frac{1}{2}e^{\varepsilon}[(t+v)e^{2\alpha\varepsilon}-(t-v)e^{-2\alpha\varepsilon}]$
1.2	$G(u)=(u^{2\alpha}+1)/(u^{2\alpha}-1)$	$\xi=2\alpha((u^{2\alpha}+1)/(u^{2\alpha}-1))$	$x^*=x+\ln\lvert(u^{2\alpha}e^{4\alpha\varepsilon}-1)/(u^{2\alpha}-1)\rvert-2\alpha\varepsilon$
		$\tau=t+2\alpha v$	$t^*=\frac{1}{2}e^{\varepsilon}[(t+v)e^{2\alpha\varepsilon}+(t-v)e^{-2\alpha\varepsilon}]$
		$\eta=2u$	$u^*=ue^{2\varepsilon}$
		$\phi=2\alpha t+v$	$v^*=\frac{1}{2}e^{\varepsilon}[(t+v)e^{2\alpha\varepsilon}-(t-v)e^{-2\alpha\varepsilon}]$
2	$G(u)=\tan(\alpha\ln u)$	$\xi=-2\alpha\tan(\alpha\ln u)$	$x^*=x+\ln\lvert\cos(\alpha(\ln u+2\varepsilon))/\cos(\alpha\ln u)\rvert$
		$\tau=t-2\alpha v$	$t^*=e^{\varepsilon}[t\cos 2\alpha\varepsilon-v\sin 2\alpha\varepsilon]$
		$\eta=2u$	$u^*=ue^{2\varepsilon}$
		$\phi=2\alpha t+v$	$v^*=e^{\varepsilon}[v\cos 2\alpha\varepsilon+t\sin 2\alpha\varepsilon]$
3	$G(u)=(\ln u)^{-1}$	$\xi=2(\ln u)^{-1}$	$x^*=x+\ln\lvert 1+2\varepsilon/\ln u\rvert$
		$\tau=t+2v$	$t^*=e^{\varepsilon}(2v\varepsilon+t)$
		$\eta=2u$	$u^*=ue^{2\varepsilon}$
		$\phi=v$	$v^*=ve^{\varepsilon}$
4	$G(u)=\tan u$	$\xi=-\tan u$	$x^*=x+\ln\lvert\cos(u+\varepsilon)/\cos u\rvert$
		$\tau=-v$	$t^*=t\cos\varepsilon-v\sin\varepsilon$
		$\eta=1$	$u^*=u+\varepsilon$
		$\phi=t$	$v^*=v\cos\varepsilon+t\sin\varepsilon$
5.1	$G(u)=\tanh u$	$\xi=\tanh u$	$x^*=x+\ln((e^{2(u+\varepsilon)}+1)/(e^{2u}+1))-\varepsilon$
		$\tau=v$	$t^*=\frac{1}{2}((t+v)e^{\varepsilon}+(t-v)e^{-\varepsilon})$
		$\eta=1$	$u^*=u+\varepsilon$
		$\phi=t$	$v^*=\frac{1}{2}((t+v)e^{\varepsilon}-(t-v)e^{-\varepsilon})$
5.2	$G(u)=\coth u$	$\xi=\coth u$	$x^*=x+\ln\lvert(e^{2(u+\varepsilon)}-1)/(e^{2u}-1)\rvert-\varepsilon$
		$\tau=v$	$t^*=\frac{1}{2}((t+v)e^{\varepsilon}+(t-v)e^{-\varepsilon})$
		$\eta=1$	$u^*=u+\varepsilon$
		$\phi=t$	$v^*=\frac{1}{2}((t+v)e^{\varepsilon}-(t-v)e^{-\varepsilon})$
6	$G(u)=u^{-1}$	$\xi=u^{-1}$	$x^*=x+\ln\lvert 1+\varepsilon/u\rvert$
		$\tau=v$	$t^*=t+v\varepsilon$
		$\eta=1$	$u^*=u+\varepsilon$
		$\phi=0$	$v^*=v$

VI. A PHYSICAL EXAMPLE

We now specialize to the situation for the second physical example mentioned in the Introduction. Here $F(u)=\lambda G'(u)$. Consequently, in Tables I and II, we have $\beta=0$ and $\int F(u)\mathrm{d}u=G(u)$. In Table V, we give the corresponding global group for each of the six classes yielding potential symmetries of the scalar equation (1).

After translations and scalings of u and G, class 5 includes bounded monotonic stress tensor functions $G(u)=\alpha\tanh(\beta u+\gamma)$ for arbitrary constants α, β, γ, and δ.

VII. FURTHER DISCUSSION

In this paper we found new symmetries for equivalent telegraph equations (1)–(4). These symmetries can be used to find families of solutions from any given solution and to construct invariant solutions from the invariants of the symmetries or through the direct method discussed in Ref. 3.

In future papers, we will find conservation laws for (1)–(4) through techniques introduced in Refs. 16–18. Systems (1)–(4) are not self-adjoint. Hence a symmetry of (1)–(4) does not yield a conservation law through Noether's theorem.

For the physical case $F(u)=G'(u)$, the second equation in the potential system (2) becomes

$$(e^x v)_t = (e^x G(u))_x.$$

This leads to another equivalent potential system

$$v_x = u_t, \quad w_x = e^x v, \quad w_t = e^x G(u).$$

The problem of finding equivalent potential systems for a given PDE is considered in Refs. 4 and 5.

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