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J. Math. Anal. Appl. 339 (2008) 1199–1209

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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Solution and asymptotic/blow-up behaviour of a class of nonlinear dissipative systems

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Received 4 June 2007

Available online 27 July 2007

Submitted by Goong Chen

Abstract

In this paper, we consider a three-parameter class of Liénard type nonlinear dissipative systems of the form $\ddot{x} + (b + 3kx)\dot{x} + k^2x^3 + bkx^2 + \lambda x = 0$. Since such dissipative systems admit an eight-parameter Lie group of point transformations, it follows that there exists a (complex) point transformation mapping such a system into the free particle system $\ddot{x} = 0$. Normally, such an explicit point transformation cannot be found. Here we find such an explicit point transformation through exploiting the group properties of the determining equations that lead to it. Consequently, we obtain the explicit general solution of such dissipative systems. Moreover, we completely characterize the asymptotic and/or finite time blow-up behaviour of such systems in terms of their three parameters and initial data.

Keywords: Nonlinear oscillator; Blow-up behaviour; Lane–Emden equation; Liénard system; Symmetry

1. Introduction

Liénard type nonlinear systems of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.1}$$

and their generalizations are widely used in applications in the context of nonlinear oscillations. A class of systems (1.1) with polynomial nonlinearities (corresponding to truncated power series expansions of $f(x)$, $g(x)$) is given by

$$\ddot{x} + (A + Bx)\dot{x} + Cx^3 + Dx^2 + Ex + F = 0. \tag{1.2}$$

In this paper, we consider the five-parameter subclass of systems (1.2) corresponding to the situation when $C = \frac{1}{9}B^2$. Through scalings and translations in x, it is easy to show that this five-parameter class of systems is equivalent to the three-parameter class of systems given by

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0022-247X/\$ – see front matter
doi:10.1016/j.jmaa.2007.06.076

$$\ddot{x} + (b + 3kx)\dot{x} + k^2x^3 + bkx^2 + \lambda x = 0, \tag{1.3}$$

where $b, k,$ and λ are arbitrary constants.

Equation (1.2) includes a transformed version of the Lane–Emden equation [1,2]

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d\theta(R)}{dR}\right) + \theta^n(R) = 0, \tag{1.4}$$

which is a Poisson equation for the gravitational potential of a self-gravitating, spherically symmetric polytropic fluid. In this model, pressure and density are related by a polytropic equation $P = \rho^{1+1/n}$. Equation (1.3) studied in the current paper includes the Lane–Emden equation (1.4) when $n = -3$ (see Ref. [2]). Equation (1.3) can also be considered as a cubic anharmonic oscillator acted upon by a strong nonlinear damping type force.

A particular subclass of systems (1.3) with $b = 0$, given by

$$\ddot{x} + 3kx\dot{x} + k^2x^3 + \lambda x = 0, \tag{1.5}$$

was studied in Refs. [3–7] and found to possess unusual dynamical properties. An explicit general solution of Eq. (1.5) was found in [8] for the case $\lambda = 0$. In [3,5,6], a rather specialized method was used to obtain a general solution of (1.5) for $\lambda \neq 0$ in an indirect manner. In [5,7], based on a method appearing in [9], it was also shown that the general solution of (1.5) is obtained through use of a nonlocal transformation that maps (1.5) into a constant coefficient linear third-order ODE.

In [6], the general solution of (1.3) was obtained indirectly through use of nonlocal transformations associated with the modified Prelle–Singer method [10]. In the present paper, through exploiting the group properties of (1.3), we show how to directly obtain the general solution of (1.3).

It is well known that

(1) any second-order nonlinear ODE can be mapped into a second-order linear ODE by some contact transformation;
(2) a second-order nonlinear ODE can be mapped into a second-order linear ODE by some point transformation (not necessarily real even if the nonlinear ODE is real) if and only if it admits a maximal eight-parameter Lie group of point transformations;
(3) any second-order linear ODE can be mapped into the free particle equation by some point transformation.

But in each of cases (1)–(3), existence of such transformations does not mean that one is able to construct them. In fact, one is rarely successful in any such constructions.

It is straightforward (see Section 4) to show that a Liénard type nonlinear system (1.1) admits an eight-parameter Lie group of point transformations, i.e., eight point symmetries, if and only if it is of the form (1.3). It follows that there exists a (complex) point transformation $X = X(x,t)$, $T = T(x,t)$ mapping (1.3) into the free particle equation

$$\frac{d^2X}{dT^2} = 0. \tag{1.6}$$

In Section 2, we show that the determining equations for the mapping of (1.3) into (1.6) yield $X(x,t) = f(t)T(x,t) + g(t)$, and that such a mapping exists for any pair of functions $(f(t), g(t))$ satisfying a coupled system of nonlinear fourth-order ODEs. By inspection, one sees that this coupled system admits five point symmetries. An invariant solution of this coupled system, resulting from these point symmetries, yields a particular mapping of (1.3) to (1.6). In turn, we show how the general complex solution of (1.6) yields the general real solution of (1.3). In Section 3, we analyze the general solution and obtain its asymptotic and/or blow-up behaviour in terms of the values of its three parameters and initial data. In Section 4, we give some concluding remarks.

2. The mapping

Since an ODE of the form (1.3) admits eight point symmetries, it follows that there exists an invertible point transformation (real or complex)

$$X = X(x,t), \qquad T = T(x,t) \tag{2.1}$$

that maps *any* solution of the nonlinear ODE (1.3) into a solution of the free particle equation (1.6). Such a mapping yields the general solution of the nonlinear ODE (1.3). Normally, one is unable to find such an explicit mapping when it exists, but here we show that this can be accomplished as follows.

Suppose the mapping (2.1) satisfies ODE (1.6) for any solution $x(t)$ of ODE (1.3). Then, by direct calculation, it follows that such a mapping exists if and only if $X(x,t)$, $T(x,t)$ satisfy the bilinear system of four PDEs given by

$$X_{xx}T_x - T_{xx}X_x = 0, \tag{2.2}$$

$$2(X_{tx}T_x - T_{tx}X_x) + X_{xx}T_t - T_{xx}X_t = 0, \tag{2.3}$$

$$(3kx+b)(T_xX_t - X_xT_t) + 2(X_{tx}T_t - T_{tx}X_t) + X_{tt}T_x - T_{tt}X_x = 0, \tag{2.4}$$

$$\left(k^2x^3 + bkx^2 + \lambda x\right)(T_xX_t - X_xT_t) + X_{tt}T_t - T_{tt}X_t = 0. \tag{2.5}$$

It is easy to see that the general solution of (2.2) yields

$$X(x,t) = f(t)T(x,t) + g(t), \tag{2.6}$$

for an arbitrary pair of functions $(f(t), g(t))$. After substituting (2.6) into the remaining equtions (2.3)–(2.5), one can show that a mapping exists if and only if $(f(t), g(t))$ satisfy the coupled system of nonlinear ODEs

$$f^{(4)} = \frac{12(\dddot{f})^2\dot{g} - 2(6\dot{f}\dddot{g} + 9\ddot{f}\ddot{g} - K_1\dot{f}\dot{g})\dddot{f} + 2(9\ddot{f}^2 - K_1\dot{f}^2)\dddot{g} - 3K_1\ddot{f}^2\dot{g} + (3K_1\ddot{g} - K_2\dot{g})\dot{f}\ddot{f} + K_2\dot{f}^2\ddot{g}}{9(\ddot{f}\dot{g} - \dot{f}\ddot{g})}, \tag{2.7}$$

$$g^{(4)} = \frac{12\dot{f}(\dddot{g})^2 - 2(6\dddot{f}\dot{g} + 9\ddot{f}\ddot{g} - K_1\dot{f}\dot{g})\dddot{g} + 2\dddot{f}(9\ddot{g}^2 - K_1\dot{g}^2) - 3K_1\dot{f}\ddot{g}^2 + (3K_1\ddot{f} - K_2\dot{f})\dot{g}\ddot{g} + K_2\ddot{f}\dot{g}^2}{9(\dot{f}\ddot{g} - \ddot{f}\dot{g})}, \tag{2.8}$$

where $K_1 = b^2 - 3\lambda$, $K_2 = 2b^3 - 9\lambda b$, $f^{(4)} = d^4f/dt^4$, and $g^{(4)} = d^4g/dt^4$.

In terms of a solution of (2.7), (2.8), it is straightforward to show that

$$T(x,t) = \frac{2\dddot{f}\dot{g}^2 - 2\dot{f}\dot{g}\dddot{g} - \ddot{f}(3\dot{g}\ddot{g} - (3kx+b)\dot{g}^2) + 3\dot{f}\ddot{g}^2 - (3kx+b)\dot{f}\dot{g}\ddot{g}}{2\dot{f}^2\dddot{g} - 2\dot{f}\dddot{f}\dot{g} - (3\dot{f}\ddot{f} - (3kx+b)\dot{f}^2)\ddot{g} + 3\ddot{f}^2\dot{g} - (3kx+b)\dot{f}\ddot{f}\dot{g}}. \tag{2.9}$$

In order to find an invertible mapping of (1.3) to (1.6), it is not necessary to find the general solution of the nonlinear system (2.7), (2.8). It is sufficient to find a particular solution of this system. Such a solution can be found as an invariant solution from an admitted point symmetry of the coupled system (2.7), (2.8) [11–13]. By inspection, one sees that the coupled system of ODEs (2.7), (2.8) admits the three-parameter $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ group of point transformations

$$\begin{aligned} f^* &= \varepsilon_1 f, \\ g^* &= \varepsilon_2 g, \\ t^* &= t + \varepsilon_3. \end{aligned} \tag{2.10}$$

The invariant form for a resulting invariant solution of (2.7), (2.8) (see [11–13] for details) is given by

$$f(t) = L_1e^{\alpha t}, \qquad g(t) = L_2e^{\beta t} \tag{2.11}$$

for arbitrary nonzero constants $L_1, L_2, \alpha \neq \beta$.

In view of the homogeneity of (2.7), (2.8), without loss of generality, $L_1 = L_2 = 1$ for such an existing solution. After substitution of (2.11) into (2.7), (2.8), it is easy to see that such an invariant solution of (2.11) exists if and only if (α, β) solve the symmetrical system of algebraic equations

$$\begin{aligned} &\left(6\lambda - 2b^2 + 6\beta^2\right)\alpha - 3\beta^3 + \left(b^2 - 3\lambda\right)\beta + 2b^3 - 9\lambda b = 0, \\ &\left(6\lambda - 2b^2 + 6\alpha^2\right)\beta - 3\alpha^3 + \left(b^2 - 3\lambda\right)\alpha + 2b^3 - 9\lambda b = 0. \end{aligned} \tag{2.12}$$

Any nontrivial solution $\alpha \neq \beta$ of (2.12) will lead to a mapping of (1.3) to (1.6). The solutions to (2.12) assume a different form depending on the sign of the quantity $\lambda - b^2/4$. (If β turns out to be zero, then the invariance of (2.7), (2.8), under the obvious larger five-parameter group that includes translations in f and g, leads to the invariant form given by $f(t)=e^{\alpha t}$, $g(t)=t$ and a corresponding set of algebraic equations for α, in order to obtain an invariant solution of (2.7), (2.8). This is indeed the case when $\lambda = b^2/4$.)

Three cases arise.

Case 1 *($\lambda - \frac{b^2}{4} > 0$).* Let $\omega = \sqrt{\lambda - \frac{b^2}{4}}$. A particular solution of (2.12) is given by

$$\alpha = -\frac{b}{2} + i\omega, \qquad \beta = 2i\omega. \tag{2.13}$$

Correspondingly, we obtain the complex mapping

$$\begin{aligned} T(x,t) &= -\frac{8\omega kx}{(b+2kx+2i\omega)(2\omega+ib)} e^{\frac{1}{2}(b+2i\omega)t}, \\ X(x,t) &= -\frac{2\omega kx - i(b^2+4\omega^2+2bkx)}{(b+2kx+2i\omega)(2\omega+ib)} e^{2i\omega t} \end{aligned} \tag{2.14}$$

of the given nonlinear ODE (1.3) into the free particle ODE (1.6). The general solution of (1.6) is $X(T) = AT + B$, which yields

$$X\big(x(t),t\big) = AT\big(x(t),t\big) + B, \tag{2.15}$$

for arbitrary complex constants A and B.

Solving (2.15) for $x(t)$, we obtain the general complex solution

$$x(t) = \frac{i}{2k} \frac{4\lambda(e^{2i\omega t} - B)}{(2\omega - ib)e^{2i\omega t} + B(2\omega+ib) - 4A\omega e^{(\frac{b}{2}+i\omega)t}} = x_1(t) + ix_2(t) \tag{2.16}$$

of the nonlinear dissipative system (1.3). Here $x_1(t) = \operatorname{Re} x(t)$, $x_2(t) = \operatorname{Im} x(t)$.

By construction, the solution (2.16) includes all solutions of ODE (1.3) for arbitrary complex constants A and B. Moreover, this solution must include the general real solution of ODE (1.3) that contains two arbitrary real constants. In particular, the general real solution is found by selecting the values of the complex constants A and B so that $x_2(t) = 0$. Consequently,

$$A = \frac{\lambda}{kM} e^{\frac{b}{2\omega}\phi} e^{-i\delta}, \qquad B = e^{-2i\delta},$$

where M and δ are arbitrary real constants. This yields the general real solution

$$x(t) = x_1(t) = \frac{\sqrt{\lambda} M \sin(\omega t + \delta)}{\sqrt{\lambda}\omega e^{\frac{b}{2\omega}(\omega t - \delta)} - kM \sin(\omega t + \delta + \phi)} \tag{2.17}$$

of ODE (1.3). Here $\phi = \tan^{-1} \frac{2\omega}{b}$.

Case 2 *($\lambda - \frac{b^2}{4} < 0$).* Let $\Omega = \sqrt{\frac{b^2}{4} - \lambda}$.

In this case a particular solution of (2.12) is given by

$$\alpha = \frac{1}{2}(\Omega - b), \qquad \beta = \Omega. \tag{2.18}$$

The resulting mapping

$$\begin{aligned} T(x,t) &= \frac{8\Omega kx}{(b+2kx+2\Omega)(b-2\Omega)} e^{\frac{1}{2}t(b+2\Omega)}, \\ X(x,t) &= -\frac{2(b+2\Omega)kx + 4\lambda}{(b+2kx+2\Omega)(b-2\Omega)} e^{2\Omega t} \end{aligned} \tag{2.19}$$

is real and maps the nonlinear ODE (1.3) into the free particle equation (1.6). The general real solution of ODE (1.3) is

$$x(t) = \frac{2}{k}\frac{\lambda(e^{2\Omega t} - B)}{Ae^{\frac{t}{2}(b+2\Omega)} + B(b - 2\Omega) - e^{2\Omega t}(b + 2\Omega)} \tag{2.20}$$

for arbitrary real constants A and B.

Case 3 *($\lambda - \frac{b^2}{4} = 0$).* Here, the general solution of the corresponding dissipative system (1.3) can be obtained by taking the appropriate limit of either solution (2.17) or the solution (2.20) as $\Omega \to 0$. More directly, we can proceed as follows. In order to obtain an invertible mapping, we seek a solution of (2.7), (2.8) in the form $f(t) = e^{\alpha t}$, $g(t) = t$. Then $\alpha = -b/2$. The mapping is found from (2.9) and (2.6) to be

$$\begin{aligned} T(x,t) &= \frac{4kx}{b(2kx+b)}e^{\frac{1}{2}bt}, \\ X(x,t) &= \frac{bt(b+2kx)+4kx}{b(2kx+b)}. \end{aligned} \tag{2.21}$$

The point transformation (2.21) maps the given nonlinear ODE (1.3) into the free particle ODE (1.6). Correspondingly, the general solution of ODE (1.3) is

$$x(t) = \frac{b^2}{2k}\frac{(t - D_1)}{D_2 e^{\frac{bt}{2}} - b(t - D_1) - 2} \tag{2.22}$$

for arbitrary real constants D_1 and D_2.

3. Analysis of the general solution

According to the Liénard theorem [14], a Liénard system (1.1) has a stable limit cycle solution if

- $f(x)$ is even and $g(x)$ is odd;
- $f(x) < 0$ for $0 < x < a$; $f(a) = 0$; $f(x) > 0$ for $x > a$;
- $g(x)$ satisfies the Lipschitz condition and monotonically increases.

A stronger sufficient condition for the existence of a stable limit cycle solution (a "generalized Liénard theorem") was presented in [15] and does not require $f(x)$ and $g(x)$ to be even and odd. One can show that for any choice of parameters (b, k, λ), the nonlinear ODE (1.3) satisfies neither of these above sufficiency conditions.

In this section, we analyze the behaviour of the solutions of the nonlinear ODE (1.3) in terms of its parameters (b, k, λ) and initial data. In particular, for any values of the parameters (b, k, λ) in (1.3), and any posed initial data, only the following types of behaviour will occur.

1. Solution stabilization in infinite time, i.e. $\lim_{t\to\infty} x(t) = \text{const}$.
2. Finite-time blowup, i.e. $\lim_{t\to t^*} x(t) = \infty$, $0 < t^* < \infty$.
3. Solution is periodic for a domain of initial data when $b = 0$.

3.1. Equilibrium points

The equilibrium solutions of the nonlinear ODE (1.3): $x(t) = x = \text{const.}$ satisfy the cubic equation

$$k^2x^3 + bkx^2 + \lambda x = 0. \tag{3.1}$$

The number of real roots of (3.1) is different for each of the three cases considered in Section 2.

- Case 1 ($\lambda - \frac{b^2}{4} > 0$): one equilibrium solution $x_1^e = 0$;
- Case 2 ($\lambda - \frac{b^2}{4} < 0$): three equilibrium solutions $x_1^e = 0$, $x_{2,3}^e = \frac{-b\pm 2\Omega}{2k}$;
- Case 3 ($\lambda - \frac{b^2}{4} = 0$): two equilibrium solutions $x_1^e = 0$, $x_2^e = -\frac{b}{2k}$.

Below we separately analyze Cases 1–3. In particular, we show that

1. Periodic behaviour can only occur in Case 1, and here only when $b=0$.
2. For *any* set of parameters (b,k,λ), there exist initial data for which the solution blows up in finite time.
3. In every case, the equilibrium solution(s) are either stable or unstable, depending on the parameter b. In Cases 2 and 3, there is always one unstable equilibrium point. Stable and unstable points exchange their roles when the sign of the parameter b changes.

3.2. *Case 1:* $\lambda - \frac{b^2}{4} > 0$

3.2.1. $b=0$ *(periodic or blow-up behaviour)*

From the form of the general solution $x(t)$ (2.17), (2.20) and (2.22), in Cases 1–3, respectively, it is evident that bounded periodic behaviour only occurs when $b=0$, $\lambda=\omega^2>0$. In this case, the nonlinear ODE (1.3) is equivalent to the two-parameter nonlinear oscillator (1.5), and its general solution is given by

$$x(t)=\frac{\omega M\sin(\omega t+\delta)}{\omega^2-kM\cos(\omega t+\delta)},\quad \omega=\sqrt{\lambda}, \tag{3.2}$$

for arbitrary constants M,δ.

For the initial data $x(0)=x_0$, $\dot{x}(0)=x_1$, one finds that the constants

$$M(x_0,x_1)=\pm\omega\frac{\sqrt{(\omega x_0)^2+\xi^2}}{|\omega^2+k\xi|},\qquad \tan\delta(x_0,x_1)=\frac{\omega x_0}{\xi},\quad \xi=x_1+kx_0^2=\text{const.}$$

The solution (3.2) is periodic for any initial data (x_0,x_1) for which the denominator of (3.2) is never zero, namely, in the case $\omega^2>|kM|$, or in terms of (x_0,x_1),

$$\frac{\omega|\omega^2+k\xi|}{|k|\sqrt{(\omega x_0)^2+\xi^2}}>1. \tag{3.3}$$

For initial data (x_0,x_1) that do not satisfy condition (3.3), the solution blows up (i.e., becomes infinite) at time $t=t^*$ satisfying

$$\cos\bigl(\omega t^*+\delta(x_0,x_1)\bigr)=\frac{\omega^2}{kM(x_0,x_1)}. \tag{3.4}$$

3.2.2. $b\neq 0$

From the form of the general solution (2.17), it follows that

(i) The equilibrium solution $x_1^e=0$ is stable if and only if $b>0$.
(ii) For $b>0$, for certain initial data the solution approaches the equilibrium solution $x^e(t)=0$, whereas for all other initial data, the solution blows up in finite time.
(iii) For $b<0$, for arbitrary initial data (x_0,x_1), the solution blows up in finite time.

In terms of initial data $x(0)=x_0$, $\dot{x}(0)=x_1$, the arbitrary constants in the general solution (2.17) are given by

$$\tan\delta(x_0,x_1)=\frac{2\omega x_0}{2(x_1+kx_0^2)+bx_0},\qquad M(x_0,x_1)=\lambda\frac{\sqrt{\lambda x_0^2+(x_1+kx_0^2)(x_1+kx_0^2+bx_0)}}{\lambda+k(x_1+kx_0^2+bx_0)}e^{-\frac{b\delta(x_0,x_1)}{2\omega}}, \tag{3.5}$$

where $\lambda=\omega^2+\frac{b^2}{4}$.

The solution blows up in finite time for all initial data (x_0,x_1) for which there exists a root $t=t^*>0$ of the transcendental equation (the denominator in the general solution (2.17))

$$\sqrt{\lambda}\omega e^{\frac{b}{2\omega}(\omega t-\delta(x_0,x_1))}-kM(x_0,x_1)\sin\bigl(\omega t+\delta(x_0,x_1)+\phi\bigr)=0. \tag{3.6}$$

Such a blow-up time t^* exists for any initial data when $b<0$, and only for initial data satisfying (3.6) for some finite time $t=t^*$ when $b>0$.

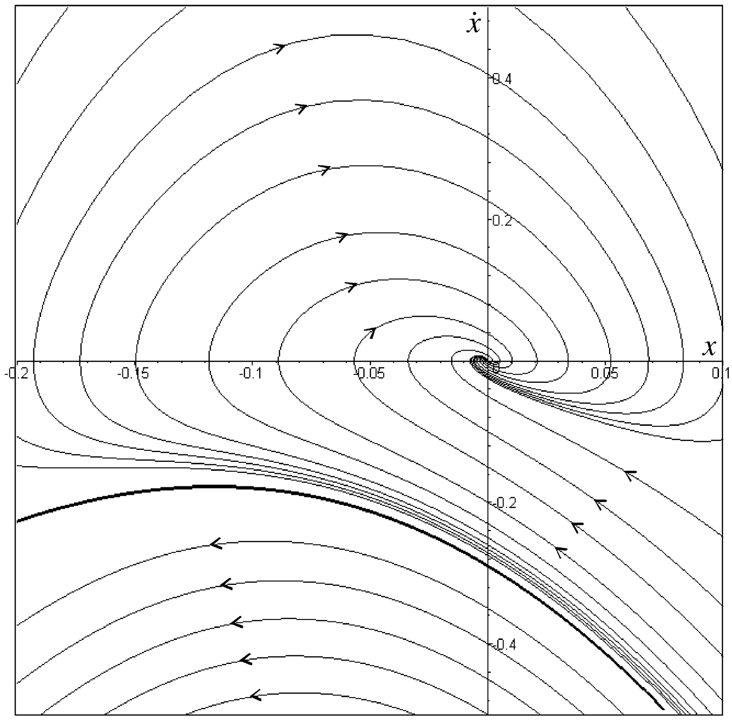

Fig. 1. Phase-plane diagram for Eq. (1.3) for Case 1: $\lambda - \frac{b^2}{4} > 0$ for parameter values $k = 10$, $b = 2$, $\lambda = 3.25$ (hence $\omega = 1.5$). The separatrix between convergent and divergent solutions is indicated by a thick line. The point $x = \dot{x} = 0$ is a stable equilibrium point.

A sample phase-plane diagram of Eq. (1.3) for Case 1: $\lambda - \frac{b^2}{4} > 0$, is shown in Fig. 1 for parameter values $k = 10$, $b = 2$, $\lambda = 3.25$ (hence $\omega = 1.5$.) Suppose $x_0 = 0$. If $x_1 < x_1^* \simeq -0.289$, the solution converges to the equilibrium point $x = \dot{x} = 0$; if $x_1 \gtrsim x_1^*$, the solution blows up in finite time. For example, if $x_1 = -0.290$, the blowup time is found from (3.6) to be $t^* \simeq 1.987$.

3.3. Case 2: $\lambda - \frac{b^2}{4} < 0$

First we note that an alternative representation for the general solution (2.20) is given by

$$x(t) = \frac{2}{k} \frac{\lambda(C_1 e^{2\Omega t} - 1)}{C_2 e^{\frac{bt}{2} + \Omega t} + (b - 2\Omega) - C_1(b + 2\Omega)e^{2\Omega t}}. \tag{3.7}$$

($C_1 = 1/B$, $C_2 = A/B$.)

In terms of initial data $x(0) = x_0$, $\dot{x}(0) = x_1$, the arbitrary constants C_1, C_2 in the general solution (3.7) are given by

$$C_1(x_0, x_1) = \frac{2x_1 + x_0(b + 2\Omega + 2kx_0)}{2x_1 + x_0(b - 2\Omega + 2kx_0)}, \qquad C_2(x_0, x_1) = \frac{2\Omega}{k} \frac{(2kx_0 + b)^2 + 4(kx_1 - \Omega^2)}{2x_1 + x_0(b - 2\Omega + 2kx_0)}. \tag{3.8}$$

3.3.1. Finite time blowup

For any choice of parameters (λ, b, k), for certain initial data, the solution blows up in finite time. Indeed, for all initial data (x_0, x_1) for which there exists a positive root of the transcendental equation (the denominator in the general solution (3.7))

$$C_2(x_0, x_1)e^{\frac{bt}{2} + \Omega t} + (b - 2\Omega) - C_1(x_0, x_1)(b + 2\Omega)e^{2\Omega t} = 0, \tag{3.9}$$

the solution blows up at time $t = t^* > 0$ which is the minimum positive root of (3.9).

Table 1
Equilibrium points of nonlinear equations (1.3) in Case 2: $\lambda - \frac{b^2}{4} < 0$

Case	$x_1^e = 0$	$x_2^e = \frac{-b+2\Omega}{2k}$	$x_3^e = -\frac{b+2\Omega}{2k}$
$\lambda > 0, b > 0$	Stable	Saddle	Unstable
$\lambda > 0, b < 0$	Unstable	Stable	Saddle
$\lambda < 0, b \gtrless 0$	Saddle	Stable	Unstable

3.3.2. Equilibrium solutions and asymptotic behaviour

As shown in Section 3.1, here Eq. (1.3) admits three equilibrium solutions $x_1^e = 0, x_{2,3}^e = \frac{-b\pm 2\Omega}{2k}$. In the phase plane $(x, \dot{x})$, one of these equilibrium points is a stable point, another is a saddle, and the remaining one is an unstable equilibrium point. The situation for the equilibrium points, depending on the parameter values, is given in Table 1.

Note that the equations for the separatrices between families of solution curves in the phase plane $(x, \dot{x})$, in each case, are found explicitly as curves corresponding to zero or infinite values for the arbitrary constants (3.8) (after substitutions $x_0 = x, x_1 = \dot{x}$.) The separatrices are three parabolas given by

$$S_1:\quad \dot{x}(x) = -x\left(\frac{b}{2} - \Omega + kx\right) = -k\big(x - x_1^e\big)\big(x - x_2^e\big),$$

$$S_2:\quad \dot{x}(x) = -x\left(\frac{b}{2} + \Omega + kx\right) = -k\big(x - x_1^e\big)\big(x - x_3^e\big),$$

$$S_3:\quad \dot{x}(x) = \frac{1}{k}\left[\Omega^2 - \left(kx + \frac{b}{2}\right)^2\right] = -k\big(x - x_2^e\big)\big(x - x_3^e\big). \tag{3.10}$$

A sample phase-plane diagram for Eq. (1.3) is shown in Fig. 2. The parameter values are $k = 1$, $b = -4$, $\lambda = 3$ (hence $\Omega = 1$). The equilibrium point $x_1^e = 0$ is unstable, $x_2^e = \frac{-b+2\Omega}{2k} = 3$ is stable, and $x_3^e = \frac{-b-2\Omega}{2k} = 1$ is a saddle point.

3.4. Case 3: $\lambda = \frac{b^2}{4}$

In terms of initial data $x(0) = x_0,\ \dot{x}(0) = x_1$, the arbitrary constants D_1, D_2 of the general solution (2.22) are given by

$$D_1(x_0, x_1) = -\frac{x_0}{x_1 + x_0(kx_0 + \frac{b}{2})}, \qquad D_2(x_0, x_1) = 2\frac{x_1 + \frac{1}{k}(kx_0 + \frac{b}{2})^2}{x_1 + x_0(kx_0 + \frac{b}{2})}. \tag{3.11}$$

3.4.1. Finite time blowup

Similarly to Case 2, for any choice of equation parameters (b, k), for certain initial data, the solution blows up in finite time. For all initial data (x_0, x_1) for which there exists a positive root of the transcendental equation (the denominator in the general solution (2.22))

$$D_2(x_0, x_1)e^{\frac{bt}{2}} - b\big(t - D_1(x_0, x_1)\big) - 2 = 0, \tag{3.12}$$

the solution blows up at time $t = t^* > 0$ which is the minimum positive root of (3.12).

3.4.2. Equilibrium solutions

As shown in Section 3.1, here Eq. (1.3) has two equilibrium solutions $x_1^e = 0, x_2^e = -\frac{b}{2k}$.

The equations for the separatrices between families of solution curves in the phase plane $(x, \dot{x})$ correspond to zero or infinite values of the arbitrary constants (3.11) (after substitutions $x_0 = x, x_1 = \dot{x}$.) The separatrices are two parabolas given by

$$S_1:\quad \dot{x}(x) = -x\left(kx + \frac{b}{2}\right) = -k\big(x - x_1^e\big)\big(x - x_2^e\big),$$

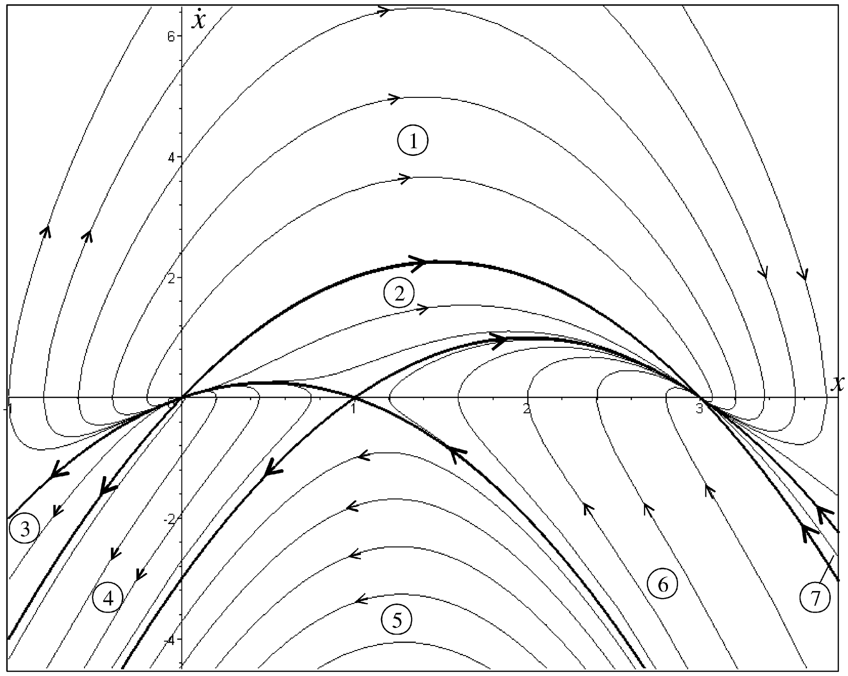

Fig. 2. Phase-plane diagram for Eq. (1.3) for Case 2: $\lambda - \frac{b^2}{4} < 0$, for parameter values $k = 1$, $b = -4$, $\lambda = 3$ (hence $\Omega = 1$). Separatrices between seven different solution families are indicated by thick lines. The point $x_1^e = 0$ is an unstable equilibrium point; the equilibrium point $x_3^e = 1$ is a saddle point; the point $x_2^e = 3$ is a stable equilibrium point. Solutions with initial data in domains 1, 2, 6 and 7 converge to the equilibrium point $x = 3$, $\dot{x} = 0$. Solutions with initial data in domains 3, 4, and 5 blow up in finite time.

$$S_2\colon \quad \dot{x}(x) = -\frac{1}{k}\left(kx_0 + \frac{b}{2}\right)^2 = -k\left(x - x_2^e\right)^2. \tag{3.13}$$

On the separatrix S_1, $D_1^{-1} = D_2^{-1} = 0$; on the separatrix S_2, $D_2 = 0$.

3.4.3. Asymptotic behaviour

Without loss of generality we assume $k > 0$ and separately consider two cases.

(i) If $b > 0$, then the exponential term yields the leading behaviour of the denominator of the general solution (2.22) for all initial data (except the separatrix S_2, where $D_2 \neq 0$). Hence the equilibrium solution $x_1^e = 0$ is *stable*. For the equilibrium solution $x_2^e = -\frac{b}{2k}$, on the separatrix S_2, one has $\dot{x} < 0$ independently of initial conditions, whereas on the separatrix S_1, $\dot{x}$ changes sign at $x_2^e = -\frac{b}{2k}$. Therefore the second equilibrium solution $x_2^e = -\frac{b}{2k}$ is a *monkey saddle* point, unstable for all solutions except for the part of the separatrix S_2 where $x > x_2^e$.

(ii) If $b < 0$, the equilibrium solution $x_1^e = 0$ is *unstable*. Using the same argument as above, we see that the equilibrium solution $x_2^e = -\frac{b}{2k}$ is stable in all directions except for the part of the separatrix S_2 where $x < x_2^e$. Hence it is a stable *monkey saddle* equilibrium point.

In Fig. 3, we give a particular example of a phase-plane diagram for Eq. (1.3) in Case 3: $\lambda = \frac{b^2}{4}$, for the parameter values $k = 2$, $b = 8$ (hence $\lambda = 16$). The equilibrium point $x_1^e = 0$ is stable, and the equilibrium point $x_2^e = -\frac{b}{2k} = -2$ is an “unstable monkey saddle.”

4. Concluding remarks

In this paper, we have used various group properties to directly solve, through (complex or real) point transformations to the (complex or real) free particle equation, a three-parameter class of real Liénard type nonlinear dissipative

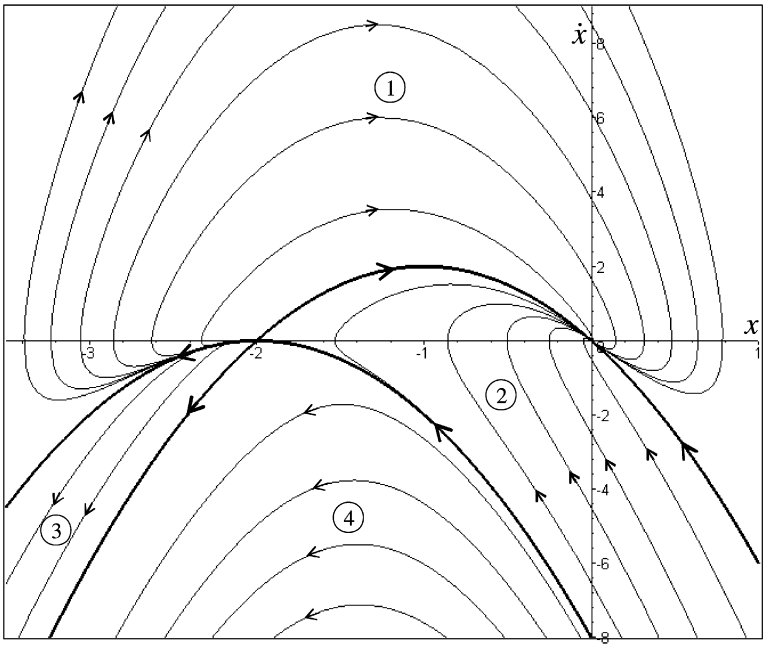

Fig. 3. Phase-plane diagram for Eq. (1.3) for Case 3: $\lambda = \frac{b^2}{4}$, for parameter values $k = 2$, $b = 8$ (hence $\lambda = 16$). Separatrices between four different solution families are indicated by thick lines. The point $x_1^e = 0$ is a stable equilibrium point; the equilibrium point $x_2^e = -2$ is an unstable "monkey saddle" point. Solutions with initial data in domains 1 and 2 converge to the equilibrium point $x = \dot{x} = 0$. Solutions with initial data in domains 3 and 4 blow up in finite time.

systems (1.3). More generally, the procedure used in this paper applies to a class of differential equations completely characterized in terms of admitted symmetries. For example, any $(1+1)$-dim linear parabolic partial differential equation that admits six nontrivial point symmetries can be mapped by an explicit point transformation into the heat equation [16].

In [17], the inverse problem was considered. In particular, it was shown that there exists an invertible point transformation $X = X(x,t)$, $T = T(x,t)$ mapping a given second-order nonlinear ODE into the free particle equation (1.6) if and only if the given ODE is of the form

$$\ddot{x} + \delta(x,t)\dot{x}^3 + \gamma(x,t)\dot{x}^2 + f(x,t)\dot{x} + g(x,t) = 0, \tag{4.1}$$

where

$$\begin{aligned}
\delta &= (X_{xx}T_x - T_{xx}X_x)/\Delta,\\
\gamma &= (X_{xx}T_t + 2X_{tx}T_x - 2T_{tx}X_x - T_{xx}X_t)/\Delta,\\
f &= (X_{tt}T_x + 2X_{tx}T_t - 2T_{tx}X_t - T_{tt}X_x)/\Delta,\\
g &= (X_{tt}T_t - T_{tt}X_t)/\Delta,
\end{aligned}$$

and the Jacobian $\Delta = X_xT_t - T_xX_t \neq 0$. Moreover, it was shown that there exists a point transformation mapping an ODE of the form (4.1) into the free particle equation if and only if its four coefficients satisfy the coupled system of nonlinear PDEs

$$\begin{aligned}
\gamma_{tt} &= -3\big(g_{xx} + (g\gamma)_x - g_t\delta\big) + 6g\delta_t + 2(f_{xt} + ff_x) - f\gamma_t,\\
\delta_{tt} &= -g\delta_x - 2g_x\delta - \frac{1}{3}(f_{xx} - f_x\gamma) + (f\delta)_t + \frac{2}{3}(\gamma_{xt} - \gamma\gamma_t).
\end{aligned} \tag{4.2}$$

Furthermore, it follows that if $\delta = \gamma = 0$, then from solving (4.2), the coefficients g and f must be of the form

$$f = a(t)x + b(t),$$

$$g = \frac{1}{9}a^2(t)x^3 + \frac{1}{3}\big(a'(t) + a(t)b(t)\big)x^2 + c(t)x + d(t) \tag{4.3}$$

in terms of four arbitrary functions $a(t), b(t), c(t), d(t)$ for the existence of a mapping of (4.1) into the free particle equation. However, in [17], explicit general solutions were given for only simple examples of ODEs of the form (4.1). These examples only included (1.3) when $b = \lambda = 0$.

Here we solve system (4.2) for the situation when (4.1) is *autonomous*, i.e., its coefficients do not depend on time t. Then one can prove that there exists a mapping by a point transformation of a given time-independent second-order nonlinear ODE to a linear ODE if and only if it is of the form (4.1) with

$$\begin{aligned} &g = g(x), \qquad f = f(x) \quad \text{arbitrary functions of } x\text{, and} \\ &\gamma = \gamma(x) = \frac{1}{3g(x)}\big(f^2(x) - 3g'(x) + A\big), \\ &\delta = \delta(x) = \frac{1}{27g^2(x)}\big(f^3(x) + 3Af(x) - 9g(x)f'(x) + B\big), \end{aligned} \tag{4.4}$$

where A and B are arbitrary constants. In particular, setting $\gamma(x) = \delta(x) = 0$ in (4.4), we observe that a standard Liénard type nonlinear system of the form (1.1) can be mapped by a point transformation to a linear ODE if and only if

$$\begin{aligned} &f(x) = 3kx + b, \\ &g(x) = k^2x^3 + bkx^2 + \lambda x + d \end{aligned} \tag{4.5}$$

(here k, b, d, λ are arbitrary constants; $d = 0$ without loss of generality), i.e. if and only if the Liénard system belongs to the class of Eqs. (1.3) studied in this paper.

The general solutions and their asymptotic/blow-up properties presented in this paper should be useful as guides when solving numerically (or through perturbation methods) ODEs of the form (1.2) when $C \neq \frac{1}{9}B^2$.

Acknowledgments

The authors acknowledge financial support from the National Sciences and Engineering Research Council of Canada and also the second author (A.F.C.) is grateful for postdoctoral support from the Pacific Institute of Mathematical Sciences (PIMS) of Canada. The work of the third author (M.S.) is sponsored by the National Board for Higher Mathematics, Government of India.

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