Framework for nonlocally related partial differential equation systems and nonlocal symmetries: Extension, simplification, and examples

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Any partial differential equation (PDE) system can be effectively analyzed through consideration of its tree of nonlocally related systems. If a given PDE system has n local conservation laws, then each conservation law yields potential equations and a corresponding nonlocally related potential system. Moreover, from these n conservation laws, one can directly construct 2^n-1 independent nonlocally related systems by considering these potential systems individually (n singlets), in pairs ($n(n-1)/2$ couplets),..., taken all together (one n-plet). In turn, any one of these 2^n-1 systems could lead to the discovery of new nonlocal symmetries and/or nonlocal conservation laws of the given PDE system. Moreover, such nonlocal conservation laws could yield further nonlocally related systems. A theorem is proved that simplifies this framework to find such extended trees by eliminating redundant systems. The planar gas dynamics equations and nonlinear telegraph equations are used as illustrative examples. Many new local and nonlocal conservation laws and nonlocal symmetries are found for these systems. In particular, our examples illustrate that a local symmetry of a k-plet is not always a local symmetry of its "completed" n-plet ($k<n$). A new analytical solution, arising as an invariant solution for a potential Lagrange system, is constructed for a generalized polytropic gas. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2349488]

I. INTRODUCTION

For any given system of partial differential equations (PDEs), one can systematically construct an extended tree of nonlocally related potential systems and subsystems.[1] All systems within a tree have the same solution set as the given system.

The analysis of a system of PDEs through consideration of nonlocally related systems in an extended tree can be of great value. In particular, using this approach, through Lie's algorithm one can systematically calculate nonlocal symmetries (which in turn are useful for obtaining new exact solutions from known ones), construct invariant and nonclassical solutions, as well as obtain linearizations, etc. (Examples are found in Ref. 1.) Perhaps more importantly, as all such related systems contain all solutions of the given system, any general method of analysis (qualitative, numerical, perturbation, conservation laws, etc.) considered for a given PDE system may be tried again on any nonlocally related potential system or subsystem. In this way, new results may be obtained for any method of analysis that is not coordinate-dependent as the systems within a tree are related in a nonlocal manner.

In Ref. 1, a tree construction algorithm is described. First, local conservation laws for the given system are found (through the direct construction method (DCM) or other method).[2–4] For each conservation law, one or several potentials are introduced.[5] Consequently, a potential system

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is obtained. Next, for each potential system, its conservation laws are computed, and further potential systems are constructed. This procedure terminates when no more new conservation laws are found. After potential systems are determined, for each potential system, new subsystems may be generated when one is able to reduce the number of dependent variables (including a reduction after a point transformation of dependent and independent variables of a potential system in a tree). At any step, all locally related potential systems and subsystems are excluded from the tree.

In this article we further extend the tree construction algorithm presented in Ref. 1. In particular, if a given system of PDEs has n conservation laws, one can directly construct 2^n-1 independent nonlocally related systems by considering their corresponding potential systems individually (n singlets), in pairs ($n(n-l)/2$ couplets),..., taken all together (one n-plet). In turn, any one of these 2^n-1 systems could lead to the discovery of new nonlocal symmetries and/or nonlocal conservation laws of the given PDE system. Moreover, such nonlocal conservation laws could yield further nonlocally related systems and subsystems as described earlier. Hence, for a given system of PDEs, the construction of its tree of nonlocally related PDE systems through our extended tree framework can be complex. Most importantly, we introduce and prove a theorem that simplifies this construction to find such extended trees by eliminating redundant systems. The work presented in this paper also simplifies and extends to within an algorithmic framework the heuristic approaches presented in Refs. 9 and 10.

This article gives a comprehensive analysis of trees of nonlocally related systems for classes of constitutive functions, including a systematic search of corresponding nonlocal symmetries and nonlocal conservation laws. In particular, new nonlocal symmetries and new conservation laws are found for planar gas dynamics (PGD) equations and nonlinear telegraph (NLT) equations, extending work in Refs. 6–10, respectively, and in references therein. Moreover, we extend and simplify the tree construction framework presented in Ref. 1 through further elimination of redundant systems. In a related work,[11] for a class of diffusion-convection equations, Popovych and Ivanova[11] completely classified its potential conservation laws and, correspondingly, constructed (hierarchical) trees of inequivalent potential systems.

This article is organized as follows. In Sec. II, we review the DCM for finding conservation laws for a given system of PDEs. We show how a related potential system arises from each local conservation law of the given system and, further, how to construct the corresponding 2^n-1 nonlocally related systems for a given system of PDEs with n local conservation laws. As examples, we consider systems of PGD equations. We find local conservation laws and corresponding nonlocally related systems for the PGD system in Lagrangian coordinates.

In Sec. III, we prove a fundamental theorem on finding conservation laws of PDE systems. In particular, for *any* given PDE system $\mathbf{F}$ with two independent variables (x and t) with precisely n local conservation laws, we show that from consideration of all combinations of the n corresponding *potential systems of PDEs* arising from the given system, no nonlocal conservation laws can be obtained for $\mathbf{F}$ through potential systems arising from multipliers that depend only on x and t. In particular, for such multipliers, all conservation laws of potential systems must be linear combinations of the n local conservation laws of the given system $\mathbf{F}$. Consequently, for such multipliers, all further potential systems are equivalent to all possible couplets, triplets,$\ldots$, n-plets of potential systems obtained from a given system $\mathbf{F}$—a total of 2^n-1 systems for consideration. Hence for a given PDE system $\mathbf{F}$, in order to find additional inequivalent potential systems as well as nonlocal conservation laws for $\mathbf{F}$, it is necessary to seek conservation laws through multipliers having an essential dependence on dependent variables. The fundamental theorem is also shown to hold for PDE systems with any number of independent variables.

In Sec. IV, as a prototypical example, we consider NLT equations. We give a complete classification of local conservation laws arising from multipliers that are functions of independent and dependent variables. As a consequence, we find five new local conservation laws arising from three distinguished cases. We then use the simplified procedure introduced in Sec. III to construct corresponding trees of nonlocally related PDE systems. Nonlocal symmetries are found for corresponding NLT systems with constitutive functions involving power law nonlinearities, including all nonlocal symmetries found in Ref. 6 as well as a new one. Moreover, six new nonlocal conservation laws are constructed for such power law NLT equations through a search of multipliers (which have an essential dependence on potential variables) for the potential systems arising from its conservation laws.

In Sec. V, we consider PGD equations, with a generalized polytropic equation of state, in Lagrangian coordinates. We give the point symmetry classification of the seven potential systems resulting from its three local conservation laws. Two new nonlocal symmetries are found which arise as point symmetries for only one of these potential systems (a couplet). We observe that these new nonlocal symmetries also arise as point symmetries of a subsystem of the Lagrange system and give the symmetry classification of this subsystem. This yields one more new nonlocal symmetry of the PGD equations. We consider invariant solutions that essentially arise from new nonlocal symmetries.

In Sec. VI, we summarize the new results presented in this article. In particular, we outline the procedure to construct a tree of nonlocally related PDE systems for a given PDE system.

In this work, a recently developed package GeM for MAPLE[12] is used for automated symmetry and conservation law analysis and classifications.

II. CONSTRUCTION OF CONSERVATION LAWS AND NONLOCALLY RELATED PDE SYSTEMS

A. Direct construction method for finding conservation laws

We first present the DCM to find the conservation laws for a general PDE system.

Let $\mathbf{G}\{\mathbf{x},\mathbf{u}\}=0$ be a system of m partial differential equations

$$\mathbf{G}\{\mathbf{x},\mathbf{u}\} = 0: \begin{cases} G_1\{\mathbf{x},\mathbf{u}\} = 0 \\ \vdots \\ G_m\{\mathbf{x},\mathbf{u}\} = 0 \end{cases} \tag{2.1}$$

with M independent variables $\mathbf{x}=(x^1,\ldots,x^M)$, and N dependent variables $\mathbf{u}=(u^1,\ldots,u^N)$. Let $\partial^l\mathbf{u}$ denote the set of all partial derivatives of $\mathbf{u}$ of order l.

A set of multipliers $\{\Lambda_k(\mathbf{x},\mathbf{U},\partial\mathbf{U},\ldots,\partial^l\mathbf{U})\}_{k=1}^m$ yields a conservation law

$$\Lambda_k(\mathbf{x},\mathbf{u},\partial\mathbf{u},\ldots,\partial^l\mathbf{u})G_k\{\mathbf{x},\mathbf{u}\} = D_i\Phi^i(\mathbf{x},\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u}) = 0 \tag{2.2}$$

of system (2.1) if and only if the linear combination $\Lambda_k(\mathbf{x},\mathbf{U},\partial\mathbf{U}\ldots,\partial^l\mathbf{U})G_k\{\mathbf{x},\mathbf{U}\}$ is annihilated by the *Euler operators*

$$E_{U^s}\frac{\partial}{\partial U^s} - D_i\frac{\partial}{\partial U^s_i} + \cdots + (-1)^j D_{i_1}\cdots D_{i_j}\frac{\partial}{\partial U^s_{i_1\ldots i_j}} + \cdots, \tag{2.3}$$

i.e., the N determining equations

$$E_{U^s}(\Lambda_k(\mathbf{x},\mathbf{U},\partial\mathbf{U},\ldots,\partial^l\mathbf{U})G_k\{\mathbf{x},\mathbf{U}\}) = 0, \quad s = 1,\ldots,N, \tag{2.4}$$

must hold for an arbitrary set of functions $\mathbf{U}=(U^1,\ldots,U^N)$. Here and for the rest of this article, we assume summation over a repeated index.

After solving the determining equations (2.4) and finding a set of multipliers $\{\Lambda_k(\mathbf{x},\mathbf{U},\partial\mathbf{U},\ldots,\partial^l\mathbf{U})\}_{k=1}^m$ that yield a conservation law, one can obtain the fluxes $\Phi^i(\mathbf{x},\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u})$ by using integral formulas arising from homotopy operators (see Refs. 2 and 3).

A conservation law $D_i\Phi^i(\mathbf{x},\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u})=0$ is called *trivial* if its fluxes are of the form $\Phi^i=M^i+H^i$, where M^i and H^i are smooth functions such that M^i vanishes on the solutions of the system (2.1), and $D_iH^i\equiv 0$. Two conservation laws $D_i\Phi^i[\mathbf{u}]=0$ and $D_i\Psi^i[\mathbf{u}]=0$ are *equivalent* if $D_i(\Phi^i[\mathbf{u}]-\Psi^i[\mathbf{u}])=0$ is a trivial conservation law. The more general "triviality" idea is the notion of linear dependence of conservation laws. A set of conservation laws is *linearly dependent* if there exists a linear combination of them which is a trivial conservation law.

B. Construction of nonlocally related systems from local conservation laws

Case A: Two independent variables. Suppose a PDE system with two independent variables

$$\mathbf{F}\{x,t,\mathbf{u}\}=0: \quad \begin{cases} F_1\{x,t,\mathbf{u}\}=0, \\ \vdots \\ F_m\{x,t,\mathbf{u}\}=0, \end{cases} \tag{2.5}$$

possesses n local conservation laws $\{\mathcal{R}^s\}_{s=1}^n$ of the form

$$\mathcal{R}^s: \quad D_x X_s(x,t,\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u}) + D_t T_s(x,t,\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u}) = 0, \quad s=1,\ldots,n, \tag{2.6}$$

where T_s and X_s are differentiable functions of their arguments. Each conservation law $\mathcal{R}^s$ (2.6) of the system (2.5) yields a pair of potential equations of the form

$$\mathcal{P}^s: \quad \begin{cases} (v_s)_x = T_s(x,t,\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u}), \\ (v_s)_t = -X_s(x,t,\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u}). \end{cases} \tag{2.7}$$

For each conservation law (2.6), the corresponding set of potential equations $\mathcal{P}^s$ (2.7) can be *appended* to the given system $\mathbf{F}$ (2.5) to yield a nonlocally related *potential system* $\mathbf{F}_\mathbf{P}^s$. (Alternatively, if at least one of the factors of the conservation law does not vanish outside of the solution space, the potential equations $\mathcal{P}^s$ can *replace* one of the equations of the given system $\mathbf{F}$.)

From the n conservation laws (2.6), one can obtain further inequivalent nonlocally related systems, by considering not only potential systems $\mathbf{F}_\mathbf{P}^s$ arising from single conservation laws $\mathcal{R}^s$, but also couplets $\{\mathbf{F}_\mathbf{P}^i,\mathbf{F}_\mathbf{P}^j\}_{i,j=1}^n$, triplets $\{\mathbf{F}_\mathbf{P}^i,\mathbf{F}_\mathbf{P}^j,\mathbf{F}_\mathbf{P}^k\}_{i,j,k=1}^n,\ldots,$ and finally the n-plet of potential systems $\{\mathbf{F}_\mathbf{P}^1,\ldots\mathbf{F}_\mathbf{P}^n\}$. Hence one obtains as many as 2^n-1 potential systems of equations nonlocally related to $\mathbf{F}$ (2.5) through the n conservation laws (2.6).

Case B: Several independent variables. Now consider a general PDE system $\mathbf{G}$ (2.1) with $M\geq 2$ independent variables. Suppose it possesses n local conservation laws $\{\mathcal{K}^s\}_{s=1}^n$ of the form

$$\mathcal{K}^s: \quad D_i\Phi_s^i(\mathbf{x},\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u}) = 0, \quad s=1,\ldots,n, \tag{2.8}$$

with fluxes $\Phi^{(s)i}$ that are differentiable functions of their arguments. Each conservation law (2.8) yields a set of M potential equations of the form (see Refs. 5 and 14)

$$\mathcal{Q}^s: \quad \Phi_s^i = \sum_{i<j}(-1)^j\frac{\partial}{\partial x^j}v_{ij} + \sum_{j<i}(-1)^{i-1}\frac{\partial}{\partial x^j}v_{ji}, \quad i=1,\ldots,M, \tag{2.9}$$

where the potentials $\mathbf{v}=\{v_{ij}(\mathbf{x})\}$ are the $\frac{1}{2}M(M-1)$ nonrepeating components of an $M\times M$ antisymmetric tensor.

For every s, by appending potential equations $\mathcal{Q}^s$ to the given system $\mathbf{G}$ (or replacing an equation of $\mathbf{G}$ by potential equations $\mathcal{Q}^s$, whatever is appropriate), one obtains a *potential system* $\mathbf{G}_\mathbf{P}^s$ which is nonlocally related to the given system $\mathbf{G}$ (2.1).

In the same manner as for the case of two independent variables, by considering singlets, couplets, triplets, $\ldots,n$-plet of potential systems $\mathbf{G}_\mathbf{P}^s$, one can obtain as many as 2^n-1 independent PDE systems nonlocally related to the given system $\mathbf{G}$, whose solution sets are equivalent to that of $\mathbf{G}$.

We now illustrate the use of 2^n-1 independent potential systems to study symmetries of a system of polytropic gas dynamics equations.

C. Conservation laws, nonlocally related PDE systems and nonlocal symmetry analysis of planar gas dynamics equations

1. Conservation laws and nonlocally related systems

In Lagrangian mass coordinates $s=t, y=\int_{x_0}^x\rho(\xi)d\xi$, planar one-dimensional gas motion is described by the equations

TABLE I. Local conservation laws of (2.10) with $\Lambda_i=\Lambda_i(y,s)$.

CL	Multipliers $(\Lambda_1,\Lambda_2,\Lambda_3)$	Conservation law	Potential	Potential equations
(W_1)	$(1,0,0)$	$D_s(q)-D_y(v)=0$	w_1	$w_{1y}=q,\ w_{1s}=v$
(W_2)	$(0,1,0)$	$D_s(v)+D_y(p)=0$	w_2	$w_{2y}=v,\ w_{2s}=-p$
(W_3)	$(y,s,0)$	$D_s(sv+yq)+D_y(sp-yv)=0$	w_3	$w_{3y}=sv+yq,\ w_{3s}=-sp+yv$

$$\mathbf{L}\{y,s,v,p,q\}=0:\quad \begin{cases} q_s-v_y=0\\ v_s+p_y=0\\ p_s+B(p,q)v_y=0. \end{cases} \tag{2.10}$$

Here x is a Cartesian space coordinate, t is time, v is the gas velocity, $q=1/\rho$ where ρ is the gas density, and p is the gas pressure. In terms of the entropy density $S(p,q)$, the constitutive function $B(p,q)$ is given by

$$B(p,q)=\frac{S_q}{S_p}.$$

We note that system (2.10) admits the group of equivalence transformations

$$s=a_1\tilde{s}+a_4,\quad y=a_2\tilde{y}+a_5,\quad v=a_3\tilde{v}+a_6,$$

$$p=\frac{a_2a_3}{a_1}\tilde{p}+a_7,\quad q=\frac{a_1a_3}{a_2}\tilde{q}+a_8,\quad B(p,q)=\frac{a_2^2}{a_1^2}\tilde{B}(\tilde{p},\tilde{q}) \tag{2.11}$$

for arbitrary constants $a_1\ldots,a_8$ with $a_1a_2a_3\neq 0$.

We first construct the simplest conservation laws and all corresponding inequivalent potential systems for the Lagrange system (2.10). Using the DCM (Sec. II A), for an arbitrary constitutive function $B(p,1/\rho)$, we find that for multipliers of the form $\Lambda_i=\Lambda_i(y,s)$, the Lagrange system (2.10) has the conservation laws exhibited in Table I.

The potential equations that arise from the conservation law (W_1) can be used to replace the first equation of the Lagrange system (2.10); potential equations arising from the conservation law (W_2), can replace the second equation of (2.10); finally, potential equations arising from the conservation law (W_3), can equivalently replace either the first or second equation of (2.10).

The independent set of nonlocally related (potential) systems of the Lagrange system (2.10) consists of the following:

- Three singlets (potential systems involving a single nonlocal variable w_i)

$$\mathbf{LW}_1\{y,s,v,p,q,w_1\}=0:\quad \begin{cases} w_{1y}=1,\\ w_{1s}=v,\\ v_s+p_y=0,\\ p_s+B(p,q)v_y=0; \end{cases} \tag{2.12}$$

$$\mathbf{LW}_2\{y,s,v,p,q,w_2\}=0:\quad \begin{cases} q_s-v_y=0,\\ w_{2y}=v,\\ w_{2s}=-p,\\ p_s+B(p,q)v_y=0; \end{cases} \tag{2.13}$$

$$\mathbf{LW}_3\{y,s,v,p,q,w_3\}=0: \quad \begin{cases} w_{3y}=sv+yq, \\ w_{3s}=-sp+yv, \\ v_s+p_y=0, \\ p_s+B(p,q)v_y=0; \end{cases} \tag{2.14}$$

- Three couplets

$$\mathbf{LW}_1\mathbf{W}_2\{y,s,v,p,q,w_1,w_2\}=0: \quad \begin{cases} w_{1y}=q, \\ w_{1s}=v, \\ w_{2y}=v, \\ w_{2s}=-p, \\ p_s+B(p,q)v_y=0; \end{cases} \tag{2.15}$$

$$\mathbf{LW}_1\mathbf{W}_3\{y,s,v,p,q,w_1,w_3\}=0: \quad \begin{cases} w_{1y}=q, \\ w_{1s}=v, \\ w_{3y}=sv+yq, \\ w_{3s}=-sp+yv, \\ p_s+B(p,q)v_y=0; \end{cases} \tag{2.16}$$

$$\mathbf{LW}_2\mathbf{W}_3\{y,s,v,p,q,w_2,w_3\}=0: \quad \begin{cases} w_{2y}=v, \\ w_{2s}=-p, \\ w_{3y}=sv+yq, \\ w_{3s}=-sp+yv, \\ p_s+B(p,q)v_y=0; \end{cases} \tag{2.17}$$

- One triplet involving all three conservation laws:

$$\mathbf{LW}_1\mathbf{W}_2\mathbf{W}_3\{y,s,v,p,q,w_1,w_2,w_3\}=0: \quad \begin{cases} w_{1y}=q, \\ w_{1s}=v, \\ w_{2y}=v, \\ w_{2s}=-p, \\ w_{3y}=sv+yq, \\ w_{3s}=-sp+yv, \\ p_s+B(p,q)v_y=0. \end{cases} \tag{2.18}$$

The Lagrange system (2.10) has also a nonlocally related *subsystem* obtained by excluding v (See Ref. 1):

$$\underline{\mathbf{L}}\{y,s,p,q\}=0: \quad \begin{cases} q_{ss}+p_{yy}=0, \\ p_s+B(p,q)q_s=0. \end{cases} \tag{2.19}$$

2. Nonlocal symmetry analysis for polytropic gas flows

We consider the polytropic equation of state

$$B(p,q) = \gamma \frac{p}{q}.$$

Applying group analysis to the triplet potential system (2.18), for arbitrary γ, one finds the basis of the ten-dimensional point symmetry algebra admitted by the given Lagrange system (2.10):

$$X_1 = \frac{\partial}{\partial s} + w_2\frac{\partial}{\partial w_3}, \quad X_2 = \frac{\partial}{\partial y} + w_1\frac{\partial}{\partial w_3},$$

$$X_3 = s\frac{\partial}{\partial s} + v\frac{\partial}{\partial v} + 2q\frac{\partial}{\partial q} + 2w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2} + 2w_3\frac{\partial}{\partial w_3},$$

$$X_4 = \frac{\partial}{\partial v} + s\frac{\partial}{\partial w_1} + y\frac{\partial}{\partial w_2} + ys\frac{\partial}{\partial w_3}, \quad X_5 = s\frac{\partial}{\partial s} + y\frac{\partial}{\partial y} + w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2} + 2w_3\frac{\partial}{\partial w_3},$$

$$X_6 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} + w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2} + w_3\frac{\partial}{\partial w_3},$$

$$X_7 = \frac{\partial}{\partial w_1}, \quad X_8 = \frac{\partial}{\partial w_2}, \quad X_9 = \frac{\partial}{\partial w_3},$$

$$X_{10} = y^2\frac{\partial}{\partial y} + (w_2 - yv)\frac{\partial}{\partial v} + yp\frac{\partial}{\partial p} - 3yq\frac{\partial}{\partial q} + (sw_2 - w_3)\frac{\partial}{\partial w_1} + yw_2\frac{\partial}{\partial w_2} + ysw_2\frac{\partial}{\partial w_3}. \tag{2.20}$$

In particular, the operators $X_1, \ldots, X_9$ project onto point symmetries of the given Lagrange system (2.10); the operator X_{10} yields a *nonlocal symmetry* of the Lagrange system **L**.[1,10]

If $\gamma=3$, system (2.10) admits one additional symmetry[9]

$$X_{11} = s^2\frac{\partial}{\partial s} + (w_1 - sv)\frac{\partial}{\partial v} - 3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q} + sw_1\frac{\partial}{\partial w_1} + (yw_1 - w_3)\frac{\partial}{\partial w_2} + ysw_1\frac{\partial}{\partial w_3},$$

which also yields a nonlocal symmetry of the Lagrange system **L**.

If $\gamma=-1$, system (2.10) corresponds to Chaplygin gas and is linearizable, as will be shown in Sec. II C 3.

Remark 1: Among all of these constructed potential systems of **L**, symmetries $X_1 \ldots, X_{10}$ (or their projections) are obtained *simultaneously* as point symmetries only for the triplet potential system $\mathbf{LW}_1\mathbf{W}_2\mathbf{W}_3$, which in this sense is a *grand system* for the Lagrange system **L**. [All other potential systems admit the corresponding projected proper subalgebras of the Lie algebra arising from (2.20).] The practical value of such a grand system is evident—possessing the largest known symmetry group, it allows the construction of a maximal possible set of invariant solutions of the given system.

Note that it does not automatically follow that the potential system with the maximum number of potential variables is a "grand system" for determining symmetries, as is the case in this example. Counterexamples will be presented in Secs. IV and V.

3. Further conservation laws for a general constitutive function

We now look for conservation law multipliers for the Lagrange system **L** (2.10) in terms of the more general form $\Lambda_i=\Lambda_i(y,s,V,P,Q)$, $i=1,2,3$.

The solution of the conservation law determining equations (2.4) yields the following multipliers:

$$\Lambda_1 = \alpha y - \beta P + B(P,Q)\Lambda_3 + \delta,$$

$$\Lambda_2 = \alpha s + \beta V + \nu,$$

$$\Lambda_3 = \Lambda_3(y,P,Q), \tag{2.21}$$

where α, β, ν, and δ are arbitrary real constants, and $\Lambda_3(y,P,Q)$ is any solution of the PDE

$$(\Lambda_3)_Q = (B(P,Q)\Lambda_3)_P - \beta. \tag{2.22}$$

Conservation laws corresponding to $\beta=0$, $\Lambda_3=0$ and $\delta,\nu,\alpha\neq 0$ are, respectively, the conservation laws (W_1), (W_2), and (W_3) listed in Table I. Additional conservation laws arise when $\Lambda_3 \neq 0$. It is possible to show that from multipliers (2.21) only two new linearly independent conservation laws follow. The first conservation law corresponds to $\beta=1$ and represents conservation of energy. It is given by

$$\left[\left(\frac{v^2}{2} + K(p,q)\right)_s + (pv)_y\right] = 0, \tag{2.23}$$

where $K(p,q)$ is a solution of the equation $K_q(p,q)=B(p,q)K_p(p,q)-p$.

The second conservation law ($\beta=0$) defines the adiabatic process in Lagrangian coordinates:

$$(S(p,q))_s = 0, \tag{2.24}$$

where the entropy $S(p,q)$ is a solution of the equation $S_q(p,q)=B(p,q)S_p(p,q)$.

For forms of $B(p,q)$ for which the functions $K(p,q)$, $S(p,q)$ can be explicitly evaluated, the conservation laws (2.23) and (2.24), respectively, yield explicit potential systems with potentials w_4, w_5.

For the polytropic case $B(p,q)=\gamma p/q$, we find that $S(p,q)=q^{\gamma}p$. The conservation law $(S(p,q))_s=0$ (2.24) can equivalently replace the last equation of the given system **L** (2.10). This leads to the potential system

$$\mathbf{LW}_5\{y,s,v,p,q,w_5\} = 0:\begin{cases}(w_5)_y(y,s) = q^{\gamma}p,\\ (w_5)_s(y,s) = 0,\\ q_s - v_y = 0,\\ v_s + p_y = 0\end{cases} \tag{2.25}$$

Noting that $w_5(y,s)=w_5(y)$ and expressing $q=k(y)p^{-1/\gamma}$, for an arbitrary $k(y)$, we find a subsystem

$$\underline{\mathbf{LW}}_5\{y,s,v,p,k\} = 0:\begin{cases}v_y - (k(y)p^{-1/\gamma})_s = 0,\\ v_s + p_y = 0\end{cases} \tag{2.26}$$

nonlocally related to the given system **L** (2.10).

Remark 2: For the case of a Chaplygin gas $\gamma=-1$, the Lagrange PGD system **L** (2.10) is nonlinear as it stands, and *cannot* be linearized by a point transformation. But the equivalent system $\underline{\mathbf{LW}}_5$ for $\gamma=-1$ becomes *linear*. Thus in the Chaplygin gas case, the Lagrange PGD system **L** is *linearized by a nonlocal transformation*.

Remark 3: Excluding the variable v from (2.26), we see that the Lagrange polytropic PGD system is equivalent to

$$\underline{\underline{\mathbf{LW}}}_5\{y,s,p\} = 0: \quad p_{yy} + (k(y)p^{-1/\gamma})_{ss} = 0, \tag{2.27}$$

which is a nonlinear elliptic equation for $k(y)>0, \gamma>-1, \gamma\neq 0$, and a nonlinear hyperbolic equation for $k(y)>0, \gamma<-1$.

Remark 4: The solutions of (2.26) for a particular form of $k(y)$ correspond to a subset of the solutions of the given system **L** (2.10). In particular, for $k(y)=\text{const}$, the system $\underline{\underline{\mathbf{LW}}}_5$ (2.26) can be mapped into a linear system by a hodograph transformation (e.g., Ref. 18). Thus it is possible to obtain a special class of solutions of the given nonlinear system **L** (2.10) through solving this *linear PDE system*.

III. LINEAR DEPENDENCE OF CONSERVATION LAWS AND LOCAL EQUIVALENCE OF POTENTIAL SYSTEMS

For a given PDE system, its conservation laws can be constructed systematically (through the Direct Construction Method or other method[2–4]). For each conservation law, one or several potentials are introduced, and the corresponding potential system is constructed. Next, for each such nonlocally related system, its conservation laws are computed, and from these, more potentials are introduced, which in turn lead to the construction of further potential systems, etc. Together with subsystems (obtained by a reduction of the number of dependent variables for a potential system, which includes consideration of reductions after an interchange of dependent and independent variables), this systematic procedure yields an extended tree of PDE systems nonlocally related to the given one (see Ref. 1 and Sec. II B).

In this section, we present theorems which simplify the tree construction through elimination of redundant systems.

A. Linear dependence of conservation laws and tree simplification. Two-dimensional case

Definition 1: Suppose the system of PDEs (2.5) has precisely n local conservation laws. Its *general potential system* **P** is the set of 2^n-1 potential systems arising from these n local conservation laws.

We now prove the following fundamental theorem concerned with the construction of further potential systems arising from **P**.

Theorem 1: *Each conservation law of any potential system in* **P**, *arising from multipliers that depend only on x and t*, *is linearly dependent on the n local conservation laws of the given system* (2.5).

Proof: Each conservation law of any system in **P**, constructed from multipliers depending only on x and t, must be of the form

$$D_x(b^i(t,x)v_i+\beta(t,x,\mathbf{u}))+D_t(a^i(t,x)v_i+\alpha(t,x,\mathbf{u}))=0, \tag{3.1}$$

for some functions $b^i(t,x), a^i(t,x), \beta(t,x,\mathbf{u}), \alpha(t,x,\mathbf{u})$.

From the compatibility conditions for multipliers of conservation laws, we immediately obtain $D_t a^i + D_x b^i = 0$. Hence

$$\int a^i dx + \int b^i dt = f^i(t) + g^i(x), \tag{3.2}$$

for some functions $f^i(t)$ and $g^i(x)$.

Now consider a conservation law (3.1) on the solution manifold of the system in **P** that it was constructed from. We have

$$D_x[b^iv_i+\beta]+D_t[a^iv_i+\alpha]=D_x\left[b^iv_i+\beta+D_t\left(\left(\int a^idx-\int b^idt\right)v_i\right)\right]+D_t\left[a^iv_i+\alpha\right.$$
$$\left.-D_x\left(\left(\int a^idx-\int b^idt\right)v_i\right)\right]=D_x\left[\beta-(v_i)_t\int b^idt\right]+D_t\left[\alpha\right.$$
$$\left.-(v_i)_x\int a^idx\right]+D_xD_t\left[\left(\int b^idt+\int a^idx\right)v_i\right]=D_x\left[\beta\right.$$

$$
- (v_i)_t \int b^i dt\Bigg] + D_t\left[\alpha - (v_i)_x \int a^i dx\right] + D_x D_t[(f_i(t) + g_i(x))v_i]
$$

$$
= D_x\left[\beta - (v_i)_t \int b^i dt + g_i(x)(v_i)_t\right] + D_t\left[\alpha - (v_i)_x \int a^i dx + f_i(t)(v_i)_x\right]. \tag{3.3}
$$

As all derivatives of potentials v_i can be expressed in terms of local variables x, t and $\mathbf{u}$, it follows that a conservation law (3.1) is linearly dependent on local ones constructed from the given system (2.5). □

Remark 5: From Theorem 1 it follows that a conservation law of any system in **P** related to the given system (2.5), arising from multipliers that depend only on x and t, is trivial on the solution manifold of **P**.

The next theorem immediately follows from Theorem 1.

Theorem 2: *Suppose one finds the set of n local conservation laws for a given system* (2.5) *and then constructs the corresponding general potential system* **P**. *It follows that if one starts with any one of the* 2^n-1 *potential systems in* **P** *and seeks conservation laws from multipliers depending only on x and t, each of the resulting potential systems is locally equivalent to one of the* 2^n-1 *potential systems in* **P**.

B. Linear dependence of conservation laws and tree simplification. General case: $M \geq 2$ independent variables

We now consider the general case for $M \geq 2$ independent variables. Suppose the system of PDEs (2.1) has a set of n conservation laws $\{\mathcal{K}^s\}_{s=1}^n$ of the form (2.8). Each conservation law $\mathcal{K}^s$ yields a set of M potential equations $\mathcal{Q}^s$ of the form (2.9) (Sec. II B).

Definition 1: Suppose the system of PDEs (2.1) has precisely n local conservation laws of the form (2.8). Its *general potential system* **Q** is the set of 2^n-1 potential systems arising from combinations of these n local conservation laws.

The following theorems generalize Theorems 1 and 2 for the case of $M \geq 2$ independent variables.

Theorem 3: *Each conservation law of any potential system in* **Q**, *arising from multipliers that depend only on independent variables* **x**, *is linearly dependent on the n local conservation laws of the given system* (2.1).

The proof of Theorem 3 is presented in the Appendix. The following theorem holds.

Theorem 4: *Suppose one finds the set* $\{\mathcal{K}^s\}_{s=1}^n$ *of n local conservation laws for the given system* **G** (2.1), *and then constructs the corresponding general potential system* **Q**. *It follows that if one starts with any one of the potential systems in* **Q** *and seeks conservation laws from multipliers depending only on the independent variables* **x**, *each of the resulting potential systems is locally equivalent to one of the potential systems in* **Q**.

Remark 6: From Theorem 4 it follows that no new nonlocally related potential systems of a given system **G** (2.1) can arise from conservation laws constructed from known potential systems of **G** with multipliers depending only on independent variables **x**.

Remark 7: Note that for any potential system in **Q**, one can allow gauge constraints relating the potentials $\{v_{ij}(\mathbf{x})\}$. In order to find nonlocal symmetries of the given system (2.1) from point symmetries of a potential system in **Q** it is necessary to adjoin such gauge constraints.[15–17]

IV. EXTENDED TREES OF NONLOCALLY RELATED PDE SYSTEMS, NONLOCAL SYMMETRIES AND NONLOCAL CONSERVATION LAWS FOR NONLINEAR TELEGRAPH EQUATIONS

As a prototypical example, for classes of NLT equations, we use the simplified procedure introduced in Sec. III to construct trees of nonlocally related PDE systems and, as a consequence, find new nonlocal symmetries and new nonlocal conservation laws.

A. Local conservation laws for the NLT equation

We consider NLT equations of the form

$$\mathbf{U}\{x,t,u\}=0: \quad u_{tt}-(F(u)u_x)_x-(G(u))_x=0. \tag{4.1}$$

Equation (4.1) and its potential versions, including

$$\mathbf{UV}\{x,t,u,v\}=0: \quad \begin{cases} u_t-v_x=0, \\ v_t-F(u)u_x-G(u)=0, \end{cases} \tag{4.2}$$

are known to possess rich conservation law and symmetry structure for various classes of constitutive functions $F(u),G(u)$.[6–8,13] In particular, the point symmetry classification of (4.1) appears in Ref. 13; the point symmetry and local conservation law classification of (4.2) appear in Refs. 6 and 7, respectively.

Using the DCM, we now construct nontrivial linearly independent local conservation laws for the NLT equations **U** (4.1). First we note that Eq. (4.1) admits the group of equivalence transformations

$$x=a_1\tilde{x}+a_4, \quad t=a_2\tilde{t}+a_5, \quad u=a_3\tilde{u}+a_6,$$

$$F(u)=a_1^2a_2^{-2}\tilde{F}(\tilde{u}), \quad G(u)=a_1a_2^{-2}a_3\tilde{G}(\tilde{u})+a_7, \tag{4.3}$$

where $a_1,\ldots,a_7$ are arbitrary constants, $a_1a_2a_3\neq 0$. We classify the local conservation laws and point symmetries of (4.1) modulo the equivalence transformations (4.3). A multiplier of the form $A(x,t,U)$ yields a local conservation law

$$D_x(X(x,t,u,u_x,u_t))+D_t(T(x,t,u,u_x,u_t))=0$$

of (4.1) if and only if the equation

$$E_U(\Lambda(x,t,U)(U_{tt}-(F(U)U_x)_x-(G(U))_x))=0 \tag{4.4}$$

holds for an arbitrary function $U(x,t)$.

Solving determining equation (4.4), one obtains an overdetermined system of linear PDEs in terms of the unknown multiplier $\Lambda(x,t,U)$. It is easy to show that $\Lambda=\Lambda(x,t)$. Three cases are distinguished. For arbitrary functions $F(u)$ and $G(u)$, one has two conservation laws (V_1) and (V_2); for the case $G'=F$, there are two additional conservation laws (B_1) and (B_2); for the case $G=u$, there are also two additional conservation laws (C_1) and (C_2). The classification is presented in Table II. [Note that the case where G is linear in u and F=const is the linear case and hence is not considered. The case G=const (with arbitrary F) is linearizable and hence also is not considered.]

The local conservation laws (V_2), (B_3), (B_4), (C_3), and (C_4) have not previously appeared in the literature.

The following potential systems result from the conservation laws listed in Table II.

Case (a): Arbitrary $F(u),G(u)$.

$$\mathbf{UV}_1\{x,t,u,v_1\}=0: \begin{cases} v_{1x}-u_t=0, \\ v_{1t}-F(u)u_x-G(u)=0; \end{cases} \tag{4.5}$$

$$\mathbf{UV}_2\{x,t,u,v_2\}=0: \begin{cases} v_{2x}-(tu_t-u)=0, \\ v_{2t}-t(F(u)u_x+G(u))=0. \end{cases} \tag{4.6}$$

Case (b): $G'(u)=F(u),F(u)$ arbitrary. In addition to potential systems (4.5) and (4.6), here we also have

TABLE II. Local conservation laws of (4.1).

$F(u)$	$G(u)$	CL	Multipliers	T	$-X$
Arbitrary	Arbitrary	(V_1)	$\Lambda=1$	u_t	$F(u)u_x+G(u)$
		(V_2)	$\Lambda=t$	tu_t-u	$t(F(u)u_x+G(u))$
Arbitrary	$G'(u)=F(u)$	(B_3)	$\Lambda=e^x$	$e^x u_t$	$e^x F(u)u_x$
		(B_4)	$\Lambda=te^x$	$e^x(tu_t-u)$	$te^x F(u)u_x$
Arbitrary ($F(u)\neq$ const)	u	(C_3)	$\Lambda=x-\frac{t^2}{2}$	$\left(x-\frac{t^2}{2}\right)u_t+ut$	$\left(x-\frac{t^2}{2}\right)(F(u)u_x+u)-\int F(u)du$
		(C_4)	$\Lambda=xt-\frac{t^3}{6}$	$\left(tx-\frac{t^3}{6}\right)u_t-\left(x-\frac{t^2}{2}\right)u$	$\left(tx-\frac{t^3}{6}\right)(F(u)u_x+u)-t\int F(u)du$

$$\mathbf{UB}_3\{x,t,u,b_3\}=0:\begin{cases} b_{3x}-e^x u_t=0, \\ b_{3t}-e^x F(u)u_x=0; \end{cases} \tag{4.7}$$

$$\mathbf{UB}_4\{x,t,u,b_4\}=0:\begin{cases} b_{4x}-e^x(tu_t-u)=0, \\ b_{4t}-te^x F(u)u_x=0. \end{cases} \tag{4.8}$$

Case (c): $G(u)=u$, $F(u)$ arbitrary. In addition to potential systems (4.5) and (4.6), here we also have

$$\mathbf{UC}_3\{x,t,u,c_3\}=0:\begin{cases} c_{3x}-\left(\left(x-\frac{t^2}{2}\right)u_t+tu\right)=0, \\ c_{3t}-\left(\left(x-\frac{t^2}{2}\right)(F(u)u_x+u)-\int F(u)du\right)=0; \end{cases} \tag{4.9}$$

$$\mathbf{UC}_4\{x,t,u,c_4\}=0:\begin{cases} c_{4x}-\left(\left(tx-\frac{t^3}{6}\right)u_t-\left(x-\frac{t^2}{2}\right)u\right)=0, \\ c_{4t}-\left(\left(tx-\frac{t^3}{6}\right)(F(u)u_x+u)-t\int F(u)du\right)=0. \end{cases} \tag{4.10}$$

We now apply Theorem 2 to find inequivalent nonlocally related potential systems for the NLT equation (4.1). The following statements hold.

Corollary 1: *In terms of multipliers depending only on x and t, the set of locally inequivalent potential systems for the NLT equation* (4.1) *with general nonlinearities $F(u)$ and $G(u)$ is exhausted by the following PDE systems*:

- *Two potential systems* (4.5) and (4.6), *involving single potentials*;
- *One couplet* {(4.5), (4.6)}.

Corollary 2: *In terms of multipliers depending only on x and t, the set of locally inequivalent potential systems for Eq.* (4.1) *with $G'(u)=F(u)$ is exhausted by the following systems*:

- *Four potential systems* (4.5)–(4.8) *involving single potentials*;
- *Six couplets* {(4.5), (4.6)}, {(4.5), (4.7)}, {(4.5), (4.8)}, {(4.6), (4.7)}, {(4.6), (4.8)}, and {(4.7), (4.8)} *involving pairs of potentials*;
- *Four triplets* {(4.5), (4.6), (4.7)}, {(4.5), (4.6), (4.8)}, {(4.5), (4.7), (4.8)}, and {(4.6), (4.7),

TABLE III. Symmetries of the NLT equation (4.1) and its potential systems (4.5), (4.6), (4.11) for the general case (a): $F(u)=u^{\alpha}$, $G(u)=u^{\beta}(\alpha,\beta,\neq 0)$.

System	Symmetries
$\mathbf{UV}_1\mathbf{V}_2$, $\mathbf{UV}_1$, $\mathbf{UV}_2$, $\mathbf{U}$	$X_1=(\alpha-\beta+1)x\frac{\partial}{\partial x}+\left(\frac{\alpha}{2}-\beta+1\right)t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}+\frac{\alpha+2}{2}v_1\frac{\partial}{\partial v_1}+(\alpha-\beta+2)v_2\frac{\partial}{\partial v_2}$, $X_2=\frac{\partial}{\partial x}$, $X_3=\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial v_2}$, $X_4=\frac{\partial}{\partial v_1}$, $X_5=\frac{\partial}{\partial v_2}$.

(4.8)} *for combinations involving three potentials*;
- *One quadruplet* {(4.5), (4.6), (4.7), (4.8)} *involving all four potentials*.

Corollary 3: *In terms of multipliers depending only on x and t, the set of locally inequivalent potential systems for Eq.* (4.1) *with arbitrary* $F(u)$ *and* $G(u)=u$ *is exhausted by the following systems*:

- *Four potential systems* (4.5), (4.6), (4.9), and (4.10) *involving single potentials*;
- *Six couplets* {(4.5), (4.6)}, {(4.5), (4.9)}, {(4.5), (4.10)}, {(4.6), (4.9)}, {(4.6), (4.10)}, *and* {(4.9), (4.10)} *involving pairs of potentials*;
- *Four triplets* {(4.5), (4.6), (4.9)}, {(4.5), (4.6), (4.10)}, {(4.5), (4.9), (4.10)}, *and* {(4.6), (4.9), (4.10)} *for combinations involving three potentials*;
- *One quadruplet* {(4.5), (4.6), (4.9), (4.10)} *involving all four potentials*.

B. Point and nonlocal symmetry analysis of NLT equations with power nonlinearities

We now apply the results of Sec. III to seek point and nonlocal symmetries of the NLT equation (4.1) with power nonlinearities $F(u)=u^{\alpha},G(u)=u^{\beta}(\alpha,\beta\neq 0)$ by considering its locally inequivalent potential systems.

Case (a): Arbitrary power nonlinearities $F(u),G(u)$. We first consider general power nonlinearities: $F(u)=u^{\alpha}$, $G(u)=u^{\beta}$ ($\alpha,\beta\neq 0$ arbitrary constants.) In this case, the given system (4.1) has two conservation laws (V_1) and (V_2) exhibited in Table II. From Corollary 1, the set of inequivalent nonlocally related potential systems of the PDE **U** (4.1) is exhausted by the systems $\mathbf{UV}_1$ (4.5), $\mathbf{UV}_2$ (4.6), and the couplet $\mathbf{UV}_1\mathbf{V}_2$:

$$\mathbf{UV}_1\mathbf{V}_2\{x,t,u,v_1,v_2\}=0:\begin{cases} v_{1x}-u_t=0,\\ v_{1t}-F(u)u_x-G(u)=0,\\ v_{2x}-(tu_t-u)=0,\\ v_{2t}-t(F(u)u_x+G(u))=0. \end{cases} \tag{4.11}$$

Symmetry generators of the given NLT equation (4.1), its potential systems (4.5) and (4.6) and the couplet (4.11) are given in Table III.

From the form of the symmetries in Table III it follows that no nonlocal symmetries arise for systems **U** and $\mathbf{UV}_1$. The generator X_3 is a nonlocal symmetry for the system $\mathbf{UV}_2$ (i.e., the system $\mathbf{UV}_2$ is not invariant under translations in t) and a point symmetry for the other systems. All other generators define point symmetries for all systems in Table III.

Case (b): $G'(u)=F(u)$. We now consider power nonlinearities $F(u)=(\alpha+1)u^{\alpha}$, $G(u)=u^{\alpha+1}$, $\alpha\neq 0,-1$. From the equivalence relation (4.3), this case is equivalent to $F(u)=u^{\alpha}$, $G(u)=u^{\alpha+1}$.

TABLE IV. Symmetries of the potential NLT systems for case for case (b): $F(u)=(\alpha+1)u^{\alpha}$, $G(u)=u^{\alpha+1}(\alpha\neq 0,-1)$.

System	$F(u)$	$G(u)$	Symmetries
$\mathbf{UV_1V_2B_3B_4}$, $\mathbf{UV_1V_2B_3}$, $\mathbf{UV_1V_2B_4}$, $\mathbf{UV_1B_3B_4}$, $\mathbf{UV_2B_3B_4}$, $\mathbf{UV_1V_2}, \mathbf{UV_1B_3}$, $\mathbf{UV_1B_4}, \mathbf{UV_2B_3}$, $\mathbf{UV_2B_4}, \mathbf{UB_3B_4}$, $\mathbf{UV_1}, \mathbf{UV_2}$, $\mathbf{UB_3}, \mathbf{UB_4}$, $\mathbf{U}$	$(\alpha+1)u^{\alpha}$	$u^{\alpha+1}$	$Y_1=-\frac{\alpha}{2}t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}+v_2\frac{\partial}{\partial v_2}+\frac{\alpha+2}{2}v_1\frac{\partial}{\partial v_1}+\frac{\alpha+2}{2}b_3\frac{\partial}{\partial b_3}+b_4\frac{\partial}{\partial b_4}$, $Y_2=\frac{\partial}{\partial x}+b_3\frac{\partial}{\partial b_3}+b_4\frac{\partial}{\partial b_4}$, $Y_3=\frac{\partial}{\partial t}+b_3\frac{\partial}{\partial b_4}+v_1\frac{\partial}{\partial v_2}$, $Y_4=\frac{\partial}{\partial v_1}$, $Y_5=\frac{\partial}{\partial v_2}$, $Y_6=\frac{\partial}{\partial b_3}$, $Y_7=\frac{\partial}{\partial b_4}$
	$-3u^{-4}$	u^{-3}	$Y_8=t^2\frac{\partial}{\partial t}+tu\frac{\partial}{\partial u}-v_2\frac{\partial}{\partial v_1}-b_4\frac{\partial}{\partial b_3}$
$\mathbf{UV_1V_2}$	$3u^2$	u^3	$Y_9=3v_1\frac{\partial}{\partial x}+(tv_1-v_2+3u)\frac{\partial}{\partial t}-uv_1\frac{\partial}{\partial u}-v_1^2\frac{\partial}{\partial v_1}-v_1v_2\frac{\partial}{\partial v_2}$

From Corollary 2, the set of inequivalent nonlocally related potential systems of the PDE **U** (4.1) is exhausted by the potential systems $\mathbf{UV_1}$ (4.5), $\mathbf{UV_2}$ (4.6), $\mathbf{UB_3}$ (4.7), $\mathbf{UB_4}$ (4.8), their six couplets, four triplets and one quadruplet. The corresponding classification of symmetry generators is presented in Table IV.

A point symmetry of any of these potential systems, where the symmetry generator components for u, x or t have an essential dependence on at least one of the potentials v_1, v_2, b_3, b_4, is a *nonlocal* (potential) symmetry of the given NLT equation (4.1).

The case $\alpha=-2$ is not considered in Table IV as here the system $\mathbf{UV_1}$ is linearizable by a point transformation.[18,19]

The point symmetries of PDE **U** (4.1) and system $\mathbf{UV_1}$ (4.5) were completely classified in Refs. 13 and 6, respectively. In Ref. 6, many new nonlocal symmetries of **U** (4.1) for other than power nonlinearities were found from the point symmetries of corresponding $\mathbf{UV_1}$ systems.

Most importantly, from Table IV, we see that for the case when $F(u)=3u^2, G(u)=u^3$, through the potential system $\mathbf{UV_1V_2}$, we have discovered a new nonlocal symmetry Y_9 for the scalar PDE **U**.

Note that Y_3 is a nonlocal symmetry for the systems $\mathbf{UV_1V_2B_4}$, $\mathbf{UV_2B_3B_4}$, $\mathbf{UV_1B_4}$, $\mathbf{UV_2B_3}$, $\mathbf{UV_2B_4}$, $\mathbf{UV_2}$, and $\mathbf{UB_4}$, and a point symmetry for the other nine systems; Y_8 is a nonlocal symmetry for the systems $\mathbf{UV_1V_2B_3}$, $\mathbf{UV_1B_3B_4}$, $\mathbf{UV_1B_3}$, $\mathbf{UV_1B_4}$, $\mathbf{UV_2B_3}$, $\mathbf{UV_1}$, $\mathbf{UB_3}$ and a point symmetry for the other nine systems; Y_9 is a point symmetry for $\mathbf{UV_1V_2}$ and a nonlocal symmetry for the other listed 15 inequivalent systems, which include $\mathbf{UV_1V_2B_3}$, $\mathbf{UV_1V_2B_4}$, and $\mathbf{UV_1V_2B_3B_4}$!

Case (c): $F(u)=u^{\alpha}, G(u)=u$. In this case, similarly to case (b), the set of independent nonlocally related potential systems of (4.1) is exhausted by the potential systems $\mathbf{UV_1}$ (4.5), $\mathbf{UV_2}$ (4.6), $\mathbf{UC_3}$ (4.9), $\mathbf{UC_4}$ (4.10), their six couplets, four triplets and one quadruplet. The corresponding classification of symmetry generators is found in Table V. The linear cases $\alpha=0,1$ are not considered.

As the simplification of overdetermined systems of linear determining equations in classification problems involving triplets $\mathbf{UV_1C_3C_4}$, $\mathbf{UV_2C_3C_4}$ and couplets $\mathbf{UV_1C_4}$, $\mathbf{UC_3C_4}$ presented a computational difficulty, the corresponding entries in Table V are not known.

From the form of the symmetries in Table V, it follows that no nonlocal symmetries arise for systems **U** and $\mathbf{UV_1}$; Z_2 is a nonlocal symmetry for the systems $\mathbf{UV_2C_3}$, $\mathbf{UC_3}$, and $\mathbf{UC_4}$ and a point symmetry for the other listed systems; Z_3 is a nonlocal symmetry for the systems $\mathbf{UV_1V_2C_4}$, $\mathbf{UV_1C_3}$, $\mathbf{UV_2C_3}$, $\mathbf{UV_2C_4}$, $\mathbf{UV_2}$, $\mathbf{UC_3}$, and $\mathbf{UC_4}$ and a point symmetry for the other listed systems. All other generators define point symmetries for the systems listed in Table V.

TABLE V. Symmetries of the potential NLT systems for case (c): $F(u)=u^{\alpha}, G(u)=u(\alpha\neq 0,1)$.

System	Case	Symmetries
$\mathbf{UV_1V_2C_3C_4}$, $\mathbf{UV_1V_2C_3}$, $\mathbf{UV_1V_2C_4}$ $\mathbf{UV_1V_2}$, $\mathbf{UV_1C_3}$, $\mathbf{UV_2C_3}$, $\mathbf{UV_2C_4}$, $\mathbf{UV_1}, \mathbf{UV_2}$, $\mathbf{UC_3}, \mathbf{UC_4}$ $\mathbf{U}$,	$\alpha\neq-1$	$Z_1=\frac{\alpha}{2}t\frac{\partial}{\partial t}+\alpha x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}+\frac{\alpha+2}{2}v_1\frac{\partial}{\partial v_1}+v_2(a+1)\frac{\partial}{\partial v_2}+\frac{3\alpha+2}{2}c_3\frac{\partial}{\partial c_3}+(2\alpha+1)c_4\frac{\partial}{\partial c_4}$, $Z_2=\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial c_3}+v_2\frac{\partial}{\partial c_4}$, $Z_3=\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial v_2}-v_2\frac{\partial}{\partial c_3}+c_3\frac{\partial}{\partial c_4}$, $Z_4=\frac{\partial}{\partial v_1}$, $Z_5=\frac{\partial}{\partial v_2}$, $Z_6=\frac{\partial}{\partial c_3}$, $Z_7=\frac{\partial}{\partial c_4}$.
	$\alpha=-1$	$Z_8=-\frac{1}{2}t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}+\frac{1}{2}v_1\frac{\partial}{\partial v_1}-\left(t+\frac{1}{2}c_3\right)\frac{\partial}{\partial c_3}-\left(\frac{t^2}{2}+c_4\right)\frac{\partial}{\partial c_4}$ Z_2, Z_3, Z_4, Z_5, Z_6, Z_7.
$\mathbf{UV_1C_3C_4}, \mathbf{UV_2C_3C_4}$ $\mathbf{UV_1C_4}, \mathbf{UC_3C_4}$		?

C. New nonlocal conservation laws for NLT equations with power nonlinearities

In this section, new nonlocal conservation laws are constructed for NLT equations (4.1) with power nonlinearities. We use the DCM for all singlet potential systems of the NLT equation (4.1) in each of cases (a), (b), and (c), allowing multipliers to have an essential dependence on dependent variables. We obtain new conservation laws for particular classes of constitutive functions. The classification is presented in the following.

For power nonlinearities $F(u)=u^{\alpha}$, $G(u)=u^{\beta}$, the set of nonlocal conservation laws is given in Table VI.

The computations were done for all systems: $\mathbf{UV}_1$, $\mathbf{UV}_2$, $\mathbf{UB}_3$, $\mathbf{UB}_4$, $\mathbf{UC}_3$, and $\mathbf{UC}_4$. No nonlocal conservation laws were found for the $\mathbf{UC}_4$ system.

The nonlocal conservation laws for PDE **U** (4.1) arising from analysis of the system $\mathbf{UV}_1$ were first found in Ref. 6. All other nonlocal conservation laws for PDE **U** (4.1) found in Table VI are new.

The case (b) with $\alpha=-2$ is not considered in Table VI as here the system $\mathbf{UV}_1$ is linearizable by a point transformation.[18,19]

V. NONLOCAL SYMMETRY CLASSIFICATION FOR GENERALIZED POLYTROPIC GAS FLOWS

We now consider the Lagrange PGD system **L** (2.10) with a generalized polytropic equation of state

$$B(p,q)=\frac{M(p)}{q}, \quad M''(p)\neq 0. \tag{5.1}$$

To construct a corresponding tree of nonlocally related potential systems, first we search for local conservation laws with multipliers of the form

$$\Lambda_i=\Lambda_i(y,s), \quad i=1,2,3.$$

The classification with respect to the constitutive function $M(p)$ reveals no special case and thus the conservation laws listed in Table I are exhaustive. According to Theorem 2, from these conservation laws we obtain the following inequivalent potential systems for the generalized polytropic Lagrange PGD system **L** (2.10):

- Three potential systems (2.12)–(2.14) involving single potentials;
- Three couplets (2.15)–(2.17) involving pairs of potentials;

TABLE VI. Nonlocal conservation laws of (4.1).

Case	System	Subcase	Multipliers	Fluxes
(a)	$\mathbf{UV}_1$	$\beta=-1$	$\Lambda_1=x+\frac{v_1^2}{2}+\frac{u^{\alpha+2}}{\alpha+2}, \Lambda_2=uv_1$	$X=-\left(\frac{u^{\alpha+2}}{\alpha+2}+\frac{v_1^2}{6}+x\right)v_1,$ $T=\left(\frac{u^{\alpha+2}}{(\alpha+2)(\alpha+3)}+\frac{v_1^2}{2}+x\right)u.$
$F(u)=u^{\alpha}$			$\Lambda_1=v_1, \Lambda_2=u.$	$X=-\frac{u^{\alpha+2}}{\alpha+2}-\frac{v_1^2}{2},$ $T=uv_1-t.$
$G(u)=u^{\beta}$		$\alpha=-1$ $\beta=-1$	$\Lambda_1=\frac{v_1^3}{3}+2(x+u)v_1+t,$ $\Lambda_2=(v_1^2+u+2x)u.$	$X=-\frac{v_1^4}{12}-(x+u)v_1^2-tv_1-\frac{u^2}{2}-2xu,$ $T=\left(u+\frac{v_1^2}{3}\right)uv_1+2xuv_1+t(u-2x).$
			$\Lambda_1=v_1^4/12+(u+x)v_1^2+tv_1+2xu+x^2+\frac{u^2}{2},$ $\Lambda_2=\left(\frac{v_1^3}{3}+t+uv_1+2xv_1\right)u.$	$X=-\frac{v_1^5}{60}-\frac{(x+u)v_1^3}{3}-\frac{(tv_1+u^2)v_1}{2}-(2u+x)xv_1-tu,$ $T=-\frac{t^2}{2}+\left(\frac{u}{3}+v_1^2+2x\right)\frac{u^2}{2}+\frac{uv_1^4}{12}+(xv_1+t)uv_1+x^2u.$
	$\mathbf{UV}_2$	$\beta=-1$	$\Lambda_1=-\frac{v_2}{t^2}, \Lambda_2=\frac{u}{t}.$	$X=-\frac{v_2^2}{2t^2}-\frac{u^{\alpha+2}}{\alpha+2}, T=\frac{uv_2-t^2}{t}.$
(b) $F(u)=(\alpha+1)u^{\alpha}$	$\mathbf{UV}_1$	$\alpha\neq-1$ $\alpha\neq-2$	$\Lambda_1=e^x u^{\alpha+1}, \Lambda_2=e^x v_1,$	$X=-e^x u^{\alpha+1}v_1,$ $T=e^x\left(\frac{u^{\alpha+2}}{\alpha+2}+\frac{v_1^2}{2}\right).$
$G(u)=u^{\alpha+1}$	$\mathbf{UV}_2$	$\alpha=-4$	$\Lambda_1=-e^x\frac{t}{u^3}, \Lambda_2=e^x v_2.$	$X=e^x\frac{tv_2}{u^3}, T=e^x\left(\frac{t^2}{u^2}-v_2^2\right).$
	$\mathbf{UB}_3$	$\alpha\neq-1$	$\Lambda_1=-u^{\alpha+1}, \Lambda_2=e^{-x}b_3.$	$X=-u^{\alpha+1}b_3, T=e^x\frac{u^{\alpha+2}}{\alpha+2}+e^{-x}\frac{b_3^2}{2}.$
	$\mathbf{UB}_4$	$\alpha=-4$	$\Lambda_1=-\frac{t}{u^3}, \Lambda_2=e^{-x}b_4.$	$X=-\frac{tb_4}{u^3}, T=\frac{1}{2}e^{-x}b_4^2-e^x\frac{t^2}{2u^2}.$
(c) $F(u)=u^{\alpha}$ $G(u)=u$	$\mathbf{UV}_1$	$\alpha=1$	$\Lambda_1=\frac{t^4}{12}-xt^2+tv_1-\frac{u^2}{2}+x^2,$ $\Lambda_2=-\frac{t^3}{3}+t(u+2x)-v_1.$	$X=\left(\frac{v_1}{2}-xt+\frac{t^3-2tu}{6}\right)u^2$ $-\left(tv_1+\frac{t^4}{12}-xt^2+x^2\right)v_1,$ $T=-\frac{u^3}{6}+\left(\frac{t^4}{12}+x^2-xt^2+tv_1\right)u$ $+\left(2xt-\frac{v}{2}-\frac{t^3}{3}\right)v_1.$
			$\Lambda_1=\frac{t^3}{6}-xt+v_1$ $\Lambda_2=-\frac{t^2}{2}+u+x.$	$X=\left(\frac{t^2}{2}-\frac{u}{3}-x\right)u^2+\left(2xt-\frac{t^3}{3}-\frac{v_1}{2}\right)v_1,$ $T=\left(\frac{t^3}{3}-2xt\right)u+(u+2x-t^2)v_1.$
	$\mathbf{UV}_2$	$\alpha=1$	$\Lambda_1=\frac{t^2}{4}-x+\frac{v_2-x^2}{t^2},$ $\Lambda_2=t-\frac{u+2x}{t}.$	$X=\frac{u^3}{3}+\frac{2x-t^2}{2}u^2+\frac{v_2^2}{2t^2}+\frac{(t^4-4x(t^2+x))v_2}{4t^2},$ $T=-\frac{uv_2}{t}-\frac{(t^4-4x(t^2+x))u}{4t}-\frac{(2x-t^2)v_2}{t}.$
	$\mathbf{UC}_3$	$\alpha=1$	$\Lambda_1=-\frac{t^2-2x}{80}+\frac{2xt^2+5u^2}{40(t^2-2x)}+\frac{4x^3+5tc_3}{10(t^2-2x)^2},$ $\Lambda_2=\frac{3t^5-20c_3}{40(t^2-2x)^2}-\frac{t(2x+u)}{4(t^2-2x)}.$	$X=-\frac{(t^2-2x)(tu^2+2c_3)}{64}+\frac{t(u^3+3tc_3)}{48}+\frac{t^4(tu^2-10c_3)+20u^2c_3}{160(t^2-2x)}+\frac{t(t^5+5c_3)c_3}{40(t^2-2x)^2},$ $T=\frac{(t^4-4x^2)u}{64}+\frac{u^3-3t^4u-6tc_3}{96}+\frac{t(t^5+10c_3)u}{80(t^2-2x)}+\frac{(t^5+5c_3)c_3}{40(t^2-2x)^2}.$

- One triplet (2.18) involving all three potentials.

TABLE VII. Symmetries of the generalized polytropic PGD system (2.10), (5.1).

System	$M(p)$	Symmetries
$\mathbf{L}$, $\mathbf{LW}_1,\mathbf{LW}_2,\mathbf{LW}_3$, $\mathbf{LW}_1\mathbf{W}_2,\mathbf{LW}_1\mathbf{W}_3$, $\mathbf{LW}_2\mathbf{W}_3$, $\mathbf{LW}_1\mathbf{W}_2\mathbf{W}_3$	(i) Arbitrary	$Z_1=\frac{\partial}{\partial s}+w_2\frac{\partial}{\partial w_3}$, $Z_2=\frac{\partial}{\partial y}+w_1\frac{\partial}{\partial w_3}$, $Z_3=\frac{\partial}{\partial v}+s\frac{\partial}{\partial w_1}+y\frac{\partial}{\partial w_2}+sy\frac{\partial}{\partial w_3}$, $Z_4=-y\frac{\partial}{\partial y}+2q\frac{\partial}{\partial q}+v\frac{\partial}{\partial v}+w_1\frac{\partial}{\partial w_1}$, $Z_5=s\frac{\partial}{\partial s}+y\frac{\partial}{\partial y}+w_1\frac{\partial}{\partial w_1}+w_2\frac{\partial}{\partial w_2}+2w_3\frac{\partial}{\partial w_3}$, $Z_6=\frac{\partial}{\partial w_1}$, $Z_7=\frac{\partial}{\partial w_2}$, $Z_8=\frac{\partial}{\partial w_3}$,
$\mathbf{L},\mathbf{LW}_2$	(ii) $-p\ln p$	$Z_9=y\frac{\partial}{\partial y}+2p\frac{\partial}{\partial p}+\frac{2q}{\ln p}\frac{\partial}{\partial q}+v\frac{\partial}{\partial v}+2w_2\frac{\partial}{\partial w_2}$.
	(iii) $\gamma p+\alpha p^{(\gamma+1)/\gamma}$ $\gamma\neq 0,-1$	$Z_{10}=\frac{(\gamma+1)y}{2\gamma}\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}-\frac{q}{\delta p^{1/\gamma}+\gamma}\frac{\partial}{\partial q}+\frac{(\gamma-1)v}{2\gamma}\frac{\partial}{\partial v}+w_2\frac{\partial}{\partial w_2}$.
	(iv) $1+\alpha e^p$, $\alpha=\pm 1$	$Z_{11}=\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}q\frac{\partial}{\partial q}-s\frac{\partial}{\partial w_2}$, $Z_{12}=y\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}yq\frac{\partial}{\partial q}-s\frac{\partial}{\partial v}-sy\frac{\partial}{\partial w_2}$.
$\mathbf{LW}_2$	(ii) $-p\ln p$	$Z_{13}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\left(3-\frac{1}{\ln p}\right)yq\frac{\partial}{\partial q}-(yv-w_2)\frac{\partial}{\partial v}+yw_2\frac{\partial}{\partial w_2}$.
	(iii) $\gamma p+\delta p^{(\gamma+1)/\gamma}$ $\gamma\neq 0,-1$	$Z_{14}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\left(3-\frac{\delta}{\gamma}\frac{p^{1/\gamma}}{\delta p^{1/\gamma}+\gamma}\right)yq\frac{\partial}{\partial q}-(yu-w_2)\frac{\partial}{\partial v}+yw_2\frac{\partial}{\partial w_2}$.

A. Classification of point and nonlocal symmetries

The classification of point symmetries of the seven potential systems (2.12)–(2.18) [modulo the equivalence transformations (2.11)] yields Table VII of point symmetries and nonlocal symmetries for the Lagrange PGD system (2.10) with the equation of state (5.1).

From Table VII, we observe that Z_{13},Z_{14} are point symmetries for the system $\mathbf{LW}_2$ and nonlocal symmetries for all other systems, including the given system $\mathbf{L}$; $Z_9,\ldots,Z_{12}$ are point symmetries for systems $\mathbf{L}$ and $\mathbf{LW}_2$ and nonlocal symmetries for all other systems.

Most importantly, we have shown that for cases (ii) and (iii), when $M(p)=-p\ln p$ and $M(p)=\gamma p+\delta p^{(\gamma+1)/\gamma}$, respectively, through the potential system $\mathbf{LW}_2$ we have discovered *new nonlocal symmetries* Z_{13} and Z_{14} for the generalized polytropic Lagrange PGD system $\mathbf{L}$ (2.10), (5.1). Note that all other generators in Table VII project onto point symmetries of the Lagrange PGD system $\mathbf{L}$ (2.10) and thus were found from point symmetry analysis of $\mathbf{L}$ in Ref. 9.

Note that the newly discovered nonlocal symmetries Z_{13} and Z_{14} of the Lagrange system $\mathbf{L}$ (2.10), (5.1) with $M(p)=-p\ln p$ and $M(p)=\gamma p+\delta p^{(\gamma+1)/\gamma}$ project onto point symmetries of the corresponding Lagrange subsystem $\underline{\mathbf{L}}$ (2.19). In other words, the point symmetries $\underline{Z}_{13}=y^2(\partial/\partial y)+py(\partial/\partial p)-(3-(1/\ln p))yq(\partial/\partial q)$, $\underline{Z}_{14}=y^2(\partial/\partial y)+yp(\partial/\partial p)-(3-(\delta/\gamma)[p^{1/\gamma}/(\delta p^{1/\gamma}+\gamma)])yq(\partial/\partial q)$ of $\underline{\mathbf{L}}$ yield nonlocal symmetries of $\mathbf{L}$.[1] It can be shown that symmetries Z_{13} and Z_{14} also yield nonlocal symmetries of the corresponding system written in terms of Eulerian coordinates.[1] The classification of point symmetries of $\underline{\mathbf{L}}$ (2.19) yields Table VIII with respect to the equation of state given by (5.1).

From Table VIII we observe that point symmetries of the Lagrange subsystem $\underline{\mathbf{L}}$ (2.19) include all corresponding point symmetries of $\mathbf{LW}_2$, and additionally for $M(p)=3p+\delta p^{4/3}$ one *new symmetry* $\underline{Z}_{15}$ is obtained. The new symmetry $\underline{Z}_{15}$ is a nonlocal symmetry of the Lagrange system $\mathbf{L}$ (2.10) and all its potential systems (2.12)–(2.18).

TABLE VIII. Point symmetries of the subsystem $\underline{\mathbf{L}}$ (2.19) of the generalized polytropic PGD system (2.10), (5.1).

$M(p)$	Symmetries
(i) Arbitrary	$\underline{Z_1}=\frac{\partial}{\partial s},\ \underline{Z_2}=\frac{\partial}{\partial y}$, $\underline{Z_4}=-y\frac{\partial}{\partial y}+2q\frac{\partial}{\partial q},\ \underline{Z_5}=s\frac{\partial}{\partial s}+y\frac{\partial}{\partial y}$.
(ii) $-p\ln p$	$\underline{Z_9}=y\frac{\partial}{\partial y}+2p\frac{\partial}{\partial p}+\frac{2q}{\ln p}\frac{\partial}{\partial q}$, $\underline{Z_{13}}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\left(3-\frac{1}{\ln p}\right)yq\frac{\partial}{\partial q}$.
(iii) $\gamma p+\delta p^{(\gamma+1)/\gamma}$	$\underline{Z_{10}}=\frac{(\gamma+1)y}{2\gamma}\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}-\frac{q}{\delta p^{1/\gamma}+\gamma}\frac{\partial}{\partial q}$, $\underline{Z_{14}}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\left(3-\frac{\alpha}{\gamma}\frac{p^{1/\gamma}}{\delta p^{1/\gamma}+\gamma}\right)yq\frac{\partial}{\partial q}$.
$\gamma=3$	$\underline{Z_{15}}=\frac{1}{3}s^2\frac{\partial}{\partial s}-sp\frac{\partial}{\partial p}+\frac{1}{\delta p^{4/3}+3}spq\frac{\partial}{\partial q}$.
(iv) $1+\alpha e^p$	$\underline{Z_{11}}=\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}q\frac{\partial}{\partial q}$, $\underline{Z_{12}}=y\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}yq\frac{\partial}{\partial q}$.

B. Nonlocally related systems and invariant solutions

1. Construction of invariant solutions for generalized polytropic PGD equations

For any given form of the constitutive function $M(p)$, different combinations of corresponding point and nonlocal symmetry generators can be used to construct families of invariant solutions of the Lagrange system **L** (2.10). As an example, we consider the case $M(p)=-p\ln p$.

The potential system $\mathbf{LW}_2$ has the largest algebra of symmetry generators. Thus it has the largest set of invariant solutions. The algebra $\mathcal{A}$ of symmetry generators for the constitutive function of interest is spanned by projections of the eight operators $Z_1,\ldots,Z_5,Z_7,Z_9,Z_{13}$ on the space of variables $\{y,s,v,p,q,w_2\}$ of $\mathbf{LW}_2$:

$$\mathcal{A}=\text{Span}\{Z_1,Z_2,Z_3,Z_4,Z_5,Z_7,Z_9,Z_{13}\}. \tag{5.2}$$

The simplest way to find all solutions of $\mathbf{LW}_2$ invariant with respect to elements of $\mathcal{A}$ consists of two steps[20]:

1. Finding optimal systems of one-dimensional invariant subalgebras $\mathcal{A}_i\subset\mathcal{A}$ and constructing solutions invariant with respect to each subalgebra $\mathcal{A}_i$;
2. Using the transformation groups corresponding to symmetry generators in $\mathcal{A}$ to extend the set of solutions.

The solutions of the Lagrange system **L** (2.10) are obtained from solutions of the potential system $\mathbf{LW}_2$ by excluding the potential variable w_2.

Following the above procedure, we first find the optimal system of one-dimensional subalgebras of $\mathcal{A}$ (5.2) (see Ref. 20.) This optimal system consists of the invariant subalgebras given by

$$\mathcal{A}_1=Z_2+\varepsilon_1Z_3,$$

$$\mathcal{A}_2=Z_2+\varepsilon_1Z_1+\varepsilon_2Z_3,$$

$$\mathcal{A}_3=Z_4+\varepsilon_1Z_1+\varepsilon_2Z_7,$$

$$\mathcal{A}_4=Z_4+\varepsilon_1Z_1+\varepsilon_2Z_2+\varepsilon_3Z_3,$$

$$\mathcal{A}_5 = Z_4 + \varepsilon_1 Z_1 + \alpha Z_9,$$

$$\mathcal{A}_6 = Z_5 + \alpha Z_4,$$

$$\mathcal{A}_7 = Z_5 + \varepsilon_1 Z_3,$$

$$\mathcal{A}_8 = Z_{13} + \varepsilon_1 Z_1 + \varepsilon_2 Z_2 + \varepsilon_3 Z_7 + \alpha Z_9. \tag{5.3}$$

Here $\varepsilon_i = 0, \pm 1$, $\alpha \in \mathbb{R}$.

The set of all resulting invariant solutions of the potential system $\mathbf{LW}_2$ [and, consequently, corresponding solutions of the Lagrange system $\mathbf{L}$ (2.10)] is obtained from solutions invariant with respect to each of the subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_8$ by means of the group transformations corresponding to the operators $Z_1, \ldots, Z_9, Z_{13}$. These group transformations are as follows:

$$\begin{aligned}
&Z_1: \quad y' = y, \quad s' = s + \epsilon_1, \quad v' = v, \quad p' = p, \quad q' = q, \quad w_2 = w_2;\\
&Z_2: \quad y' = y + \epsilon_2, \quad s' = s, \quad v' = v, \quad p' = p, \quad q' = q, \quad w_2 = w_2;\\
&Z_3: \quad y' = y, \quad s' = s, \quad v' = v + \epsilon_3, \quad p' = p, \quad q' = q, \quad w_2' = w_2 + \epsilon_3 y;\\
&Z_4: \quad y' = e^{-\epsilon_4} y, \quad s' = s, \quad v' = v, \quad p' = p, \quad q' = a^{2\epsilon_4} q, \quad v' = w_2 + \epsilon_4 y;\\
&Z_5: \quad y' = e^{\epsilon_5} y, \quad s' = e^{\epsilon_5} s, \quad v' = v, \quad p' = p, \quad q' = q, \quad w_2' = e^{\epsilon_5} w_2,\\
&Z_7: \quad y' = y, \quad s' = s, \quad v' = v, \quad p' = p, \quad q' = q, \quad w_2' = w_2 + \epsilon_7;\\
&Z_9: \quad y' = e^{\epsilon_9} y, \quad s' = s, \quad v' = e^{\epsilon_9} v, \quad p' = e^{2\epsilon_9} p, \quad q' = (1 + 2\epsilon_9/\ln p) q, \quad w_2' = e^{2\epsilon_9} w_2;\\
&Z_{13}: \quad y' = \frac{y}{1 - \epsilon_{13} y}, \quad s' = s, \quad v' = v + \epsilon_{13}(w_2 - yv),\\
&\qquad p' = \frac{p}{1 - \epsilon_{13} y}, \quad q' = \frac{1 - \epsilon_{13} y}{\ln p} q \ln \frac{p}{1 - \epsilon_{13} y}, \quad w_2' = \frac{w_2}{1 - \epsilon_{13} y}.
\end{aligned} \tag{5.4}$$

Particular solutions of the Lagrange system $\mathbf{L}$ (2.10) are obtained as solutions invariant with respect to any linear combination of generators $Z_1, \ldots, Z_5, Z_7, Z_9, Z_{13}$, possibly transformed further by using one or more Lie groups (5.4).

2. An invariant solution from a nonlocal symmetry

For the case $M(p) = -p \ln p$, we construct a solution of the Lagrange system $\mathbf{L}$ (2.10) arising from a solution of the potential system $\mathbf{LW}_2$ (2.13) invariant with respect to the subalgebra $\mathcal{A}_8$ (5.3) with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0, \alpha = 1$, i.e., from operator

$$X = Z_{13} + \alpha Z_9 = (y^2 + y)\frac{\partial}{\partial y} + (y + 2)p\frac{\partial}{\partial p} - \left(3y - \frac{y - 2}{\ln p}\right) q\frac{\partial}{\partial q} - (yv - v - w_2)\frac{\partial}{\partial v} + (y + 2)w_2\frac{\partial}{\partial w_2}.$$

One can show that this solution of $\mathbf{L}$ (2.10) does not arise as an invariant solution of an admitted point symmetry of $\mathbf{L}$. In particular, this solution has the form

$$p(y,s) = \frac{\gamma\beta^2}{\alpha^2}\frac{y^2}{y + \alpha}(1 - \tanh^2(\beta s)),$$

$$q(y,s) = -\frac{\gamma}{(y + \alpha)^3} \ln\left[\frac{\gamma\beta^2}{\alpha^2}\frac{y^2}{y + \alpha}(1 - \tanh^2(\beta s))\right],$$

$$v(y,s) = -\frac{\gamma\beta}{\alpha^2}\frac{y(y+2\alpha)}{(y+\alpha)^2}\tanh(\beta s), \tag{5.5}$$

where α, β, and γ are arbitrary constants.

For $\alpha=1$, $\beta=1$, $\gamma=-2$, after the application of the equivalence transformation (2.11) with $a_1=a_3=1$, $a_2=a_4=a_6=0$, $a_5=-1$, $a_7=p_0$, $a_8=q_0$, this yields the solution

$$p = \tilde{p}(y,s) = p_0 - \frac{2y^2}{y+1}\frac{1}{\cosh^2 s},$$

$$q = \tilde{q}(y,s) = q_0 - \frac{2}{(y+1)^3}\ln\left[\frac{2y^2}{y+1}\frac{1}{\cosh^2 s}\right],$$

$$v = \tilde{v}(y,s) = 2\frac{y(y+2)}{(y+1)^2}\tanh s \tag{5.6}$$

of the Lagrange system **L** (2.10) for the constitutive function

$$B(p,q) = -\frac{(p_0-\tilde{p})\ln(p_0-\tilde{p})}{q_0-q}.$$

For $p_0=9, q_0=1$, the pressure $p=\tilde{p}(y,s)$, density $p=1/\tilde{q}(y,s)$ and velocity $v=\tilde{v}(y,s)$ profiles at times $s=0.1, 0.8, 1.3$ are shown in Figure 1 with thin, medium and thick lines, respectively. The solution is regular, bounded and satisfies physical conditions $p>0, \rho>0$ for all times $s \geq 0$ for the material space interval $0 \leq y \leq 5$.

VI. CONCLUDING REMARKS

In this article, we extended the procedure presented in Ref. 1 to construct a tree of nonlocally related systems for a given PDE system **G** (2.1). In summary, the extended procedure is as follows.

1. **Construction of conservation laws**. Using the DCM (Sec. II A) or other method, construct local conservation laws of the given system **G**. Note that some conservation laws can be present in the given system as it stands.
2. **Construction of potential systems**. For each of the n known conservation laws $\{\mathcal{K}^s\}_{s=1}^n$ of the given system **G**, introduce potential(s) and construct a potential system $G_{\mathbf{P}}^s (s=1,\ldots,n)$. Let $\mathcal{T}_1$ denote the set of systems that consists of the given system **G**, potential systems $\mathbf{G}_{\mathbf{P}}^s$ and all possible couplets, triplets,…, n-plets of the potential systems $\mathbf{G}_{\mathbf{P}}^s$. The tree $\mathcal{T}_1$ includes a total of 2^n inequivalent systems.
3. **Construction of subsystems**. For each system in the tree $\mathcal{T}_1$, exclude where possible, one by one, dependent variables (including exclusions following interchanges of independent and dependent variables, i.e., where an independent variable becomes a dependent variable and vice versa through a point transformation), to generate all subsystems of the systems in the tree $\mathcal{T}_1$. Eliminate subsystems that are locally related to existing systems. This yields a possibly larger tree $\mathcal{T}_2$.
4. **Continuation**. In the tree $\mathcal{T}_2$, first distinguish the systems that arise from multipliers depending only on independent variables. For each such system, use the DCM or other method to construct the conservation laws for multipliers with an essential dependence on dependent variables. Construct all combinations of further potential systems arising from these conservation laws (i.e., couplets, triplets, etc.). For the other systems in the tree $\mathcal{T}_2$, construct all possible conservation laws (these can even arise from multipliers that depend only on the independent variables) and, correspondingly, construct all combinations of further potential systems. Find all nonlocally related subsystems by reduction of dependent variables. This yields an extended tree $\mathcal{T}_3$.

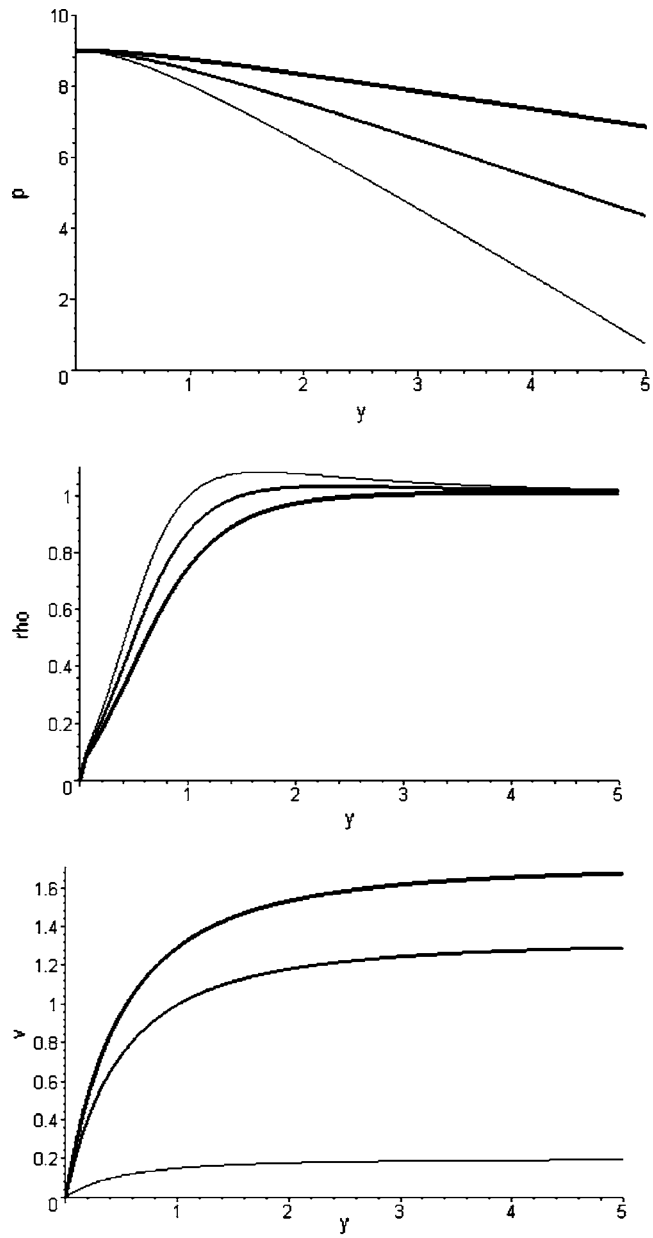

FIG. 1. Profiles of pressure p, density ρ, and velocity v at times $s=0.1$, 0.8, and 1.3.

Where possible, repeat step 4 to obtain a further tree extension (growth), etc.

The new theorem presented in Sec. III simplifies the construction of a tree of nonlocally related inequivalent systems for a given system of PDEs through elimination of redundant systems. To illustrate this theorem, as a prototypical example, we considered the nonlinear telegraph equations. Five new local conservation laws were constructed. Specializing to NLT equations with constitutive functions having power law nonlinearities, we found one nonlocal symmetry not found in Ref. 6. Further, from nonlocally related potential systems arising from new conservation laws for such NLT equations, we have found six new nonlocal conservation laws in addition to the nine nonlocal conservation laws found in Ref. 7.

For a system of planar gas dynamics equations, with a generalized polytropic equation of state, we found three new symmetries which are nonlocal for this system written in either Lagrangian or Eulerian coordinates.

It still remains a challenge to solve the overdetermined linear systems of PDEs for the symmetry classifying problems corresponding to the two couplet systems $\mathbf{UV}_1\mathbf{C}_4$, $\mathbf{UC}_3\mathbf{C}_4$ and the two triplet systems $\mathbf{UV}_1\mathbf{C}_3\mathbf{C}_4$, $\mathbf{UV}_2\mathbf{C}_3\mathbf{C}_4$ as we have been unable to solve any of these four systems.

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APPENDIX: PROOF OF THEOREM 3

Proof: Each conservation law of any k-plet potential system $\{\mathbf{G}_{\mathbf{P}}^{i_1},\ldots,\mathbf{G}_{\mathbf{P}}^{i_k}\}$ in $\mathbf{Q}$, arising from multipliers that depend only on independent variables $\mathbf{x}$, is a linear combination of terms involving potential equations in $\{\mathbf{G}_{\mathbf{P}}^{i_1},\ldots,\mathbf{G}_{\mathbf{P}}^{i_k}\}$ and, possibly, equations of the given system (2.1).

For simplicity, we prove the theorem for the case when the new conservation law is obtained as a linear combination of potential equations of a *singlet* potential system $\mathbf{G}_{\mathbf{P}}^{s}$ in $\mathbf{Q}$ arising from the given system (2.1) and a single conservation law (2.8), and involving M potential equations $\mathcal{Q}^s$ (2.9). The proof directly carries over to the case when the new conservation law involves a linear combination of potential and non-potential equations of *any k-plet potential system* in $\mathbf{Q}$, $1\le k \le n$.

A new conservation law obtained using the DCM from a set of M potential equations $\mathcal{Q}^s$ has the form

$$D_kA^k(\mathbf{x},\,\mathbf{u},\partial\mathbf{u},\,\ldots,\partial^r\mathbf{u},\mathbf{v},\partial\mathbf{v},\,\ldots,\partial^s\mathbf{v}) = \Lambda_i(\mathbf{x})\left(\sum_{i<j}(-1)^j\frac{\partial}{\partial x^j}v_{ij}(\mathbf{x}) + \sum_{j<i}(-1)^{i-1}\frac{\partial}{\partial x^{ji}}v_{ji}(\mathbf{x}) - \Phi^i(\mathbf{x},\mathbf{u},\partial\mathbf{u},\,\ldots,\partial^r\mathbf{u})\right) = 0. \tag{A1}$$

where $A^k(\mathbf{x},\mathbf{u},\mathbf{v},\partial\mathbf{u},\ldots,\partial^r\mathbf{u})$ are fluxes of the new conservation law, and $\Lambda_i(\mathbf{x})(i=1,\ldots,M)$ are multipliers. [Note that for the case of M independent variables, from a given conservation law (2.8), one obtains M potential equations (2.9). Hence, when one seeks a new conservation law, the number of multipliers is the same as the number of independent variables.]

It is evident that the dependence of fluxes of the new conservation law (A1) on the potentials $\mathbf{v}$ is as follows:

$$A^k = \sum_{i<k}(-1)^k\Lambda_i v_{ik} + \sum_{k<i}(-1)^{i-1}\Lambda_i v_{ki} + \alpha_k(\mathbf{x},\mathbf{u}). \tag{A2}$$

We substitute (A2) in the conservation law (A1), and deduce the following compatibility conditions for multipliers:

$$\frac{\partial\Lambda_q}{\partial x^p} - \frac{\partial\Lambda_p}{\partial x^q} = 0, \quad 1\le p,q\le n. \tag{A3}$$

This means the differential form $\omega_\Lambda = \Lambda_i dx^i$ is closed. A closed form is locally exact within an open domain, and hence for some sufficiently smooth $\phi(\mathbf{x}):\omega_\Lambda = d\phi(\mathbf{x})$. Equivalently $\Lambda_i = \partial\phi(\mathbf{x})/\partial x^i, \quad i=1,\ldots,M$.

We now demonstrate that the conservation law (A1) with fluxes A^k is equivalent to a conservation law whose fluxes do not contain the nonlocal variables v^{ik}, but only their derivatives.

Indeed,

$$
\begin{aligned}
D_kA^k &= D_k\left(\sum_{i<k}(-1)^k\Lambda_i v_{ik} + \sum_{k<i}(-1)^{i-1}\Lambda_i v_{ki} + \alpha_k(\mathbf{x},\mathbf{u})\right) \\
&= D_k\left(\sum_{i<k}(-1)^k\frac{\partial\phi}{\partial x^i}v_{ik} + \sum_{k<i}(-1)^{i-1}\frac{\partial\phi}{\partial x^i}v_{ki} + \alpha_k(\mathbf{x},\mathbf{u})\right) \\
&= D_k\left(\left[\sum_{i<k}(-1)^k\frac{\partial(\phi v_{ik})}{\partial x^i} + \sum_{k<i}(-1)^{i-1}\frac{\partial(\phi v_{ki})}{\partial x^i}\right]\right. \\
&\quad \left. - \phi\left[\sum_{i<k}(-1)^k\frac{\partial v_{ik}}{\partial x^i} + \sum_{k<i}(-1)^{i-1}\frac{\partial v_{ki}}{\partial x^i}\right] + \alpha^k(\mathbf{x},\mathbf{u})\right).
\end{aligned}
$$

The divergence of the flux part involving the first rectangular bracket is identically zero [see (2.8), (2.9)].

As all derivatives of potentials v_{ik} can be expressed in terms of local variables $\mathbf{x}$ and $\mathbf{u}$ on the solution manifold of $\mathbf{G}^s_{\mathbf{P}}$, it follows that the flux part involving the second rectangular bracket and $\alpha_k(\mathbf{x},\mathbf{u})$ contains only local variables of the given system (2.1). Hence the conservation law (A1) is linearly dependent on local ones constructed from the given system (2.5), and hence is trivial on the solution manifold of $\mathbf{G}^s_{\mathbf{P}}$. This concludes the proof. □

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