1. Assigned Problems

(1) In each of (a),(c),(d), determine whether or not the given Markov chain is irreducible, identify each state as recurrent or transient, and as periodic or aperiodic.

(a) The Markov chain (on six states, 0,1,...,5) with transition matrix

\[
P = \begin{bmatrix}
0 & 1/2 & 0 & 0 & 1/2 & 0 \\
1 & 0 & 0 & 1/2 & 1/2 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 2/3 & 0 & 1/3 \\
\end{bmatrix}
\]

*Hint:* Draw the transition diagram.

(b) For the Markov chain in (a), suppose the current state is 5. What is the probability that it will be in state 5 after 4 steps?

(c) SRW on \(\mathbb{Z}^d\) (no drift). (Answer for each \(d \geq 1\).)

(d) The branching process with Bin\((2, \frac{1}{2})\) offspring distribution. (Recall the definition given in class, or see Ross §4.7.)

(2) Suppose we have two urns and \(2d\) balls, of which \(d\) are black and \(d\) are white. Initially, \(d\) of the balls are placed in Urn 1, and the remainder of the balls are placed in Urn 2.

In each trial a ball is chosen uniformly at random from each of the urns, and then put back in the other urn (that is, their locations are switched).

Let \(X_0\) denote the number of black balls initially in Urn 1 and, for \(n \geq 1\), let \(X_n\) denote the number of black balls in Urn 1 after the \(n\)th trial.

(a) Explain why \((X_n)_{n \geq 0}\) is a Markov chain.

(b) Find the transition matrix \(P\) of the Markov chain.

(3) In a certain city, one out of every five sunny days is followed by a rainy day, whereas one out of every four rainy days is followed by a sunny day.

View this situation as a 2-state Markov chain.

(a) Find the transition matrix \(P\) of the Markov chain.

(b) Find the stationary distribution of the Markov chain.

(c) Determine the long run proportion of sunny days.

(4) A taxi works in Richmond and Vancouver.

From historical data, it is known that 80% of trips from Vancouver stay in Vancouver and 20% go to Richmond; whereas, 60% of trips from in Richmond go to Vancouver and 40% stay in Richmond.

A trip within Vancouver generates an average profit of $6, a trip within Richmond generates an average profit of $5, and a trip involving both cities generates an average profit of $10.

View this situation as a 2-state Markov chain.

(a) Find the transition matrix \(P\) of the Markov chain.

(b) Find the stationary distribution of the Markov chain.

(c) Determine the average profit the taxi makes per trip.
2. Required problems (but not to be handed in)

The topics on Markov chains have few problems to be handed in for marking. It is essential that you do the following problems on Markov chains prior to the final exam.
Ross §4 (same problem #’s in 10th and 11th editions): #20, 64, 67 [for (f), it is easiest to verify that \( \pi_i P_{ij} = \pi_j P_{ji} \)], 70, 76 [168].

3. Recommended problems

These provide additional practice but are not to be handed in. Starred problems have solutions in the text, and answers are given otherwise.

Ross §4 (same problem #’s in 10th and 11th editions): #1*, 16*, 18[5/9, .44440], 32*, 35[12/37, 6/37, 4/37, 3/37, 12/37], 54.