MATH 318: HOMEWORK #1

DUE FRIDAY, JANUARY 15, AT THE START OF CLASS
Late assignments will not be accepted

1. Assigned Problems

In all solutions involving permutations and combinations, be sure to briefly explain all factors arising in your solution.

1. Ross §1, #1.

2. (a) Ross §1, #4.
   (b) Suppose that $E$ and $F$ intersect and are disjoint with $G$, that is, $E \cap F \neq \emptyset$ and $(E \cup F) \cap G = \emptyset$.

(i) Draw a Venn diagram depicting the events $E, F, G$ contained in the sample space $S$. (Make sure to clearly label $E, F, G, S$.)

(ii) For each of the events discussed in (a), make a copy of the figure in (i) and shade in regions as required to depict the corresponding event. (If the event is $\emptyset$, draw an unshaded copy of the figure.)

3. A researcher requires an estimate for the number of trout in a lake. To this end, she captures 50 trout, marks each fish, and releases them into the lake. Two days later she returns to the lake and captures 80 trout, of which 16 are marked.

(a) Suppose that the lake contains $n$ trout. Find the probability $L(n)$ that 16 trout are marked in a sample of 80.

(b) Show that the function $L(n)$ is initially an increasing function of $n$ which then becomes decreasing after reaching a maximum value. Find the maximum likelihood estimate for $n$, that is, the value of $n$ which maximizes $L(n)$. Hint: Find the values of $n$ for which $L(n)/L(n-1) \leq 1$.

4. (a) Compute (with explanations) the probability that a poker hand contains

(i) (exactly) one pair ($aabcd$ with $a, b, c, d$ distinct face values). Answer: 0.4226.

(ii) (exactly) two pairs ($aabbc$ with $a, b, c$ distinct face values). Answer: 0.04754.

(b) Poker dice is played by simultaneously rolling 5 dice. Compute the probability of obtaining (exactly) two pairs ($aabbcc$ with $a, b, c$ distinct numbers). Answer: 0.2315.

5. The number of ways to place $n$ distinguishable balls in $m$ urns is $m^n$, since each ball can be placed in any one of the $m$ urns. The multinomial coefficient ${n \choose n_1, \ldots, n_m} = \frac{n!}{n_1! \cdots n_m!}$ counts the number of ways that $n_i$ balls are in urn $i$ for $i = 1, 2, \ldots, m$. Thus when each ball is randomly assigned to an urn, the probability that $n_i$ balls are in urn $i$, for each $i$, is given by $\left( \frac{n}{n_1, \ldots, n_m} m^{-n} \right)$. (Such a system is said to obey Maxwell–Boltzmann statistics.)

(a) Suppose instead that the balls are indistinguishable. (Such systems are said to obey Bose–Einstein statistics.) Note that when there are $m = 2$ urns, the number of ways of putting the $n$ balls in the 2 urns is simply $n + 1$ (since an outcome is specified by saying how many balls are in urn 1 and the possibilities are $\{0, 1, 2, \ldots, n\}$). For the case of general $m \geq 1$, find the number of ways to place $n$ indistinguishable balls in $m$ urns. Hint: This is the number of ways to arrange $m - 1$ barriers among a row of $n$ balls. For instance, for $n = 7$ and $m = 3$, the configuration with $n_1 = 0, n_2 = 2, n_3 = 5$ is $|oo|00000$. 

Date: January 8, 2016.
(b) Indistinguishable particles are said to obey Fermi–Dirac statistics if in all possible arrangements, there is at most one ball per urn, and all such arrangements occur with equal probability. For general \(m \geq n\), what is probability of any such arrangement with \(m\) urns and \(n\) particles?

(6) In this question, you investigate the birthday problem discussed in class with Octave (or MATLAB). Your code should be easy to read and adequately commented.

(a) Without doing any calculations, make a guess for the smallest value of \(n\) so that there is at least a 99% chance of (at least) two people having the same birthday in a group of \(n\) people.

(b) Write a function \(\text{birthday}(n)\) that

(i) generates a vector of \(n\) numbers uniformly distributed on the set \(\{1, 2, \ldots, 365\}\) (think of this as a list of the birthdays of \(n\) people)

(ii) returns 1 if there is at least one pair of people with the same birthday, and returns 0 otherwise.

*Hint:* Make use of the command \texttt{unidrnd(365,1,n)}. (Type \texttt{help unidrnd} for details.)

(c) For \(n = 2, 3, \ldots, 70\), run \(\text{birthday}(n)\) 1000 times and compute the proportion \(X(n)\) of times there is a matching birthday. *Hint:* Create a matrix \(B(n,i)\), where \(B(n,i)=\text{birthday}(n)\) for \(i = 1, 2, \ldots, 1000\) and put \(X(n) = 1000^{-1} \sum_{i=1}^{1000} B(n,i)\). To calculate \(X(n)\), make use of the command \texttt{sum(B(n,:))}.

(d) Recall from class that the probability that in a group of \(n\) people there is a pair of people with the same birthday is given by

\[
Y(n) = 1 - \frac{365 \cdot 364 \cdots (365 - (n - 1))}{365^n}.
\]

On a single graph, plot \(X(n)\) and \(Y(n)\) vs \(n \in [2, 70]\). (Make sure to label the figure using the commands \texttt{title}, \texttt{xlabel}, and \texttt{ylabel}.) *Hint:* To calculate \(Y(n)\) make use of the command \texttt{cumprod([365:-1:0])(n)}. (Type \texttt{help cumprod} for details.)

2. Recommended Problems

These provide additional practice but are not to be handed in. Starred problems have solutions in the text, and answers are given otherwise.

Ross §1: \#2*, 5*, 6, 7, 8, 9*, 10, 11 (\(\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}\)), 17*, 32*. 