MATH 312

Assignment 2

Solutions

1. (a) Player 2 has two options: either remove 1 or 2 matches from a pile (WLOG we can assume it is the first pile.) If Player 2 removes 1 match, then Player 1 can respond by removing a match from the other pile, so there are 2 piles with 1 match each, and we know that this is a winning position for Player 1. In the other case, Player 1 can win directly by removing all the matches in the other pile. Hence $(2, 2)$ is a winning position for Player 1.

   (b) Call a position even if its representation has an even number of 1s in each column, and odd otherwise. First, we will show that any move from an even position creates an odd position, and from an odd position there exists a move to an even position. If we show this, then it is clear that all winning positions must be even; Indeed, Player 2 always ends up in an odd position, and the final position, a pile with no matches, is even. Hence Player 2 cannot win, so Player 1 wins. Now we prove the desired result.

   Suppose you have an even position. Player 1 can only remove matches from one pile, and thus can only change the values in one row. Since Player 1 must remove at least one match, either a 1 becomes a 0, or a 0 becomes a 1. In either case, some column (namely a column containing a changed digit) now has an odd number of 1s, so the resulting position is odd. Since this didn't depend on the move, it follows that every even position leads to an odd position.

   Now suppose you are given an odd position. Let i be the leftmost column with an odd number of 1s. Since column i contains an odd number of 1s, some row has a 1 in the ith position (otherwise there are no 1s, which is even.) We will show that there is a move in that row that gives an even position. To do this, we simply replace the 1 by 0, and for every digit to the right, we change it to either 0 or 1 so that each column after the ith column has an even number of 1s. This is possible, since if a column has an even number of 1s, we change it to 0, and if it has an odd number of 1s, then we change it to 1. This value will be the amount of matches we want remaining in the pile. The final step is to check that we can reach this by removing matches. The original value of the row can be written as

   $$2^{n-i+1}k_1 + 2^{n-i} + k_2,$$

   where k_1 is an arbitrary positive integer, and $k_2 < 2^{n-i}$, and the resulting value of the pile we want is of the form

   $$2^{n-i+1}k_1 + k_3,$$

   where $k_3 < 2^{n-i}$. (k_1 remains the same because every digit to the right of the ith column is unchanged.) Therefore the resulting value of the pile will be smaller than the original value, so we can reach this position (which is even by construction.) Hence the original claim is proven, and all even positions are winning positions.

2. If n is odd, we can write $n = (n - 9) + 9$. Now, $n - 9 > 2$ since $n > 11$, and $n - 9$ is even, so $n - 9$ must be composite. 9 is also composite, so every odd $n > 11$ can be written as the sum of 2 composite numbers.

   Now suppose n is even. Then we can write $n = (n - 4) + 4$, and $n - 4$ is even (and $n - 4 > 2$ from earlier), so every even $n > 11$ can also be written as the sum of 2 composite numbers. Therefore all $n > 11$ can be written as the sum of 2 composite numbers.

3. $f(n) = n^2 + n + 41$ is not always prime for $n \in \mathbb{N}$. For example, $f(41) = 41^2$. Now, suppose $f(n)$ is an arbitrary quadratic polynomial and $f(n)$ is prime for all $n \in \mathbb{N}$. Let $g(n) = f(n+1)$. Then in particular, $g(0) = f(1) = p$ for some prime p, so we can write

$$g(n) = an^2 + bn + p$$

for some choice of $a, b \in \mathbb{Z}$. Thus for any $k \in \mathbb{N}$,

$$g(kp) = ak^2p^2 + bkp + p = p(ak^2p + bk + 1).$$

Since $g(kp) = f(kp+1)$ is prime by assumption, it follows that $g(kp) = p$ for all $k \in \mathbb{N}$, that is $g(n) - p$ has infinitely many roots. Clearly that is impossible, so the original assumption cannot be true.

4. We will prove the contrapositive statement, which says that if n is composite, so is $2^n - 1$. To that end, let $n = ab$, where $1 < a, b < n$ and $a, b \in \mathbb{Z}$. Then

$$2^{ab} - 1 = (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1).$$

Now $2^a - 1 > 1$ since $a > 1$, and $2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1 > 1$ since $b > 1$. Therefore $2^{ab} - 1$ must be composite.

The converse statement is not true. For example,

$$2^{11} - 1 = 2047 = 23 \times 89,$$

but 11 is prime.

5. We knkow that for $x, y \in \mathbb{Z}$, $\gcd(x, y) = 1$ if and only if there exist $a, b \in \mathbb{Z}$ such that $ax + by = 1$. Now,

$$7(4k + 3) - 4(7k + 5) = 28k + 21 - 28k - 20 = 1,$$

so $\gcd(7k + 5, 4k + 3) = 1$.