Math 226, 2022 : Solutions for Assignment 2

Section 10.3, Question 13

If we have

$$\overline{u}+\overline{v}+\overline{w}=\overline{0},$$

then

$$\overline{0}=\overline{0}\times\overline{v}=(\overline{u}+\overline{v}+\overline{w})\times\overline{v}=\overline{u}\times\overline{v}+\overline{v}\times\overline{v}+\overline{w}\times\overline{v}=\overline{u}\times\overline{v}+\overline{0}+\overline{w}\times\overline{v}=\overline{u}\times\overline{v}+\overline{w}\times\overline{v}.$$

It follows that

$$\overline{u}\times\overline{v}=-\overline{w}\times\overline{v}=\overline{v}\times\overline{w}.$$

A similar argument, using symmetry, gives that

$$\overline{v}\times\overline{w}=\overline{w}\times\overline{u}.$$

Section 10.3, Question 15

It's useful to do exercise 14 first, where one can work out the tetrahedron's volume in terms of the scalar triple product :

$$V=\frac{1}{6}\left|\overline{u}\cdot(\overline{v}\times\overline{w})\right|.$$

We thus have, since the tetrahedron with vertices $(1,0,0),(1,2,0),(2,2,2)$ and $(0,3,2)$ is spanned by the vectors $\overline{u}=2\overline{j}$, $\overline{u}=\overline{i}+2\overline{j}+2\overline{k}$ and $\overline{w}=-\overline{i}+3\overline{j}+2\overline{k}$,

$$V=\frac{1}{6}\left|\begin{vmatrix}0&2&0\\1&2&2\\-1&3&2\end{vmatrix}\right|=\frac{1}{6}|-8|=\frac{4}{3}\text{ cubic units.}$$

Section 10.3, Question 19

If we have

$$\overline{x}=\lambda\overline{u}+\mu\overline{v}+\nu\overline{w},$$

then

$$\overline{x}\cdot(\overline{v}\times\overline{w})=\lambda\overline{u}\cdot(\overline{v}\times\overline{w})+\mu\overline{v}\cdot(\overline{v}\times\overline{w})+\nu\overline{w}\cdot(\overline{v}\times\overline{w})=\lambda\overline{u}\cdot(\overline{v}\times\overline{w}).$$

It follows, since $\overline{u} \cdot (\overline{v} \times \overline{w}) \neq 0$, that

$$\lambda = \frac{\overline{x} \cdot (\overline{v} \times \overline{w})}{\overline{u} \cdot (\overline{v} \times \overline{w})}.$$

Using that

$$\overline{u} \cdot (\overline{v} \times \overline{w}) = \overline{v} \cdot (\overline{w} \times \overline{u}) = \overline{w} \cdot (\overline{u} \times \overline{v}),$$

symmetry implies that

$$\mu = \frac{\overline{x} \cdot (\overline{w} \times \overline{u})}{\overline{u} \cdot (\overline{v} \times \overline{w})} \quad \text{and} \quad \nu = \frac{\overline{x} \cdot (\overline{u} \times \overline{v})}{\overline{u} \cdot (\overline{v} \times \overline{w})}.$$

Section 10.3, Question 20

Since $\overline{v} \times \overline{w} \neq \overline{0}$, it follows that $(\overline{v} \times \overline{w}) \cdot (\overline{v} \times \overline{w}) \neq 0$. Using exercise 19, there must exist constants λ, μ and ν such that

$$\overline{u} = \lambda\overline{v} + \mu\overline{w} + \nu (\overline{v} \times \overline{w}).$$

Since $\overline{v} \times \overline{w}$ is perpendicular to $\overline{v}$ and $\overline{w}$,

$$\overline{u} \cdot (\overline{v} \times \overline{w}) = 0 + 0 + \nu (\overline{v} \times \overline{w}) \cdot (\overline{v} \times \overline{w})$$

Since $\overline{u} \cdot (\overline{v} \times \overline{w}) = 0$, it follows that $\nu = 0$, so that

$$\overline{u} = \lambda\overline{v} + \mu\overline{w}.$$

Section 10.3, Question 27

If we write

$$\overline{a} = -\overline{i} + 2\overline{j} + 3\overline{k} \quad \text{and} \quad \overline{b} = \overline{i} + 5\overline{j},$$

the equation becomes $\overline{a} \times \overline{x} = \overline{b}$. If this has a solution $\overline{x}$, then

$$\overline{a} \cdot \overline{b} = \overline{a} \cdot (\overline{a} \times \overline{x}) = 0,$$

contradicting the fact that $\overline{a} \cdot \overline{b} = -1 + 10 + 0 = 9$.

Section 10.4, Question 5

The plane through the points $(1,1,0)$, $(2,0,2)$, and $(0,3,3)$ has normal vector

$$\left(\bar{i}-\bar{j}+2\bar{k}\right)\times\left(\bar{i}-2\bar{j}-3\bar{k}\right)=7\bar{i}+5\bar{j}-\bar{k},$$

and hence equation (using the first point)

$$7(x-1)+5(y-1)-(z-0)=0,$$

or

$$7x+5y-z=12.$$

Section 10.4, Question 9

A plane through the line $x+y=2, y-z=3$ has equation of the form

$$x+y-2+\lambda(y-z-3)=0.$$

To have this plane perpendicular to $2x+3y+4z=5$, we require

$$2\cdot 1+(1+\lambda)\cdot 3-\lambda\cdot 4=0,$$

i.e. $\lambda=5$. The equation of the desired plane is thus $x+6y-5z=17$.