The University of British Columbia

October 5th, 2022

Midterm 1 for MATH 226 : Solutions

Closed book examination Time: 50 minutes

Last Name ____________________ **First** ________________

Signature ________________

Student Number ________________

Special Instructions:

No memory aids are allowed. No calculators. No communication or other electronic devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Midterms written in pencil will not be considered for regrading.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		12
2		12
3		6
4		8
5		6
6		6
Total		50

1. Determine whether each of the sets below is open, closed or neither. What is the boundary and interior of each set?

(a) $\{(x, y, z) \in \mathbb{R}^3 \ : \ 0 < \sqrt{x^2 + y^2} \leq 2\}$

Solution : The set is neither open nor closed. The interior is the set $0 < \sqrt{x^2 + y^2} < 2$ and the boundary is the cylinder $x^2 + y^2 = 4$ and the line $x = y = 0$.

(b) $\{(x, y, z) \in \mathbb{R}^3 \ : \ z \geq 0, \ x^2 + y^2 + z^2 \geq 3\}$

Solution : The set is closed. The interior is the set $z > 0, \ x^2 + y^2 + z^2 > 3$ and the boundary is the half-sphere $z \geq 0, \ x^2 + y^2 + z^2 = 3$, together with the set $z = 0, x^2 + y^2 \geq 3$.

2. Find the equations (any form is OK) for the line of intersection of the planes defined by $2x - y + z = -2$ and $x + 2y - z = 4$. Then find (with proof) the point on this line that is closest to the origin.

Solution : The two planes have normal vectors $\overline{n_1} = 2\overline{i} - \overline{j} + \overline{k}$ and $\overline{n_2} = \overline{i} + 2\overline{j} - \overline{k}$. We can take a direction vector $\overline{v}$ for our line to be

$$\overline{v} = \overline{n_1} \times \overline{n_2} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -\overline{i} + 3\overline{j} + 5\overline{k}.$$

To find a point on our line, let's, for example, set $z = 0$ and solve for x and y. We find that $2x - y = -2$ and $x + 2y = 4$. Solving this gives $x = 0$ and $y = 2$, so that $(x, y, z) = (0, 2, 0)$ is a point on our line of intersection. We thus get a parametric equation for the line :

$$\overline{r} = (0, 2, 0) + t(-1, 3, 5).$$

The square of the distance from a point corresponding to the parameter t to the origin is thus

$$f(t) = (0 - t)^2 + (2 + 3t)^2 + (0 + 5t)^2 = 35t^2 + 12t + 4.$$

We wish to minimize this function, so we note that

$$f'(t) = 70t + 12 = 0$$

precisely when $t = t_0 = -12/70 = -6/35$. The point on the line closest to the origin thus has coordinates

$$(x, y, z) = (-t_0, 2 + 3t_0, 5t_0) = (6/35, 52/35, -6/7).$$

3. Find the volume of the parallelepiped in $\mathbb{R}^3$ spanned by the vectors $\bar{i}-\bar{j}+4\bar{k}$, $3\bar{j}-\bar{k}$ and $\bar{i}+\bar{j}$.

Solution :

$$V = \left| \begin{vmatrix} 1 & -1 & 4 \\ 0 & 3 & -1 \\ 1 & 1 & 0 \end{vmatrix} \right| = |1\cdot 1 - (-1)\cdot 1 + 4\cdot(-3)| = |-10| = 10.$$

4. Write the vector $\overline{w} = 3\bar{i} - 5\bar{j} + \sqrt{3}\ \bar{k}$ as a sum of two vectors $\overline{w} = \overline{u} + \overline{v}$, where $\overline{u}$ is parallel to the plane $x + 2y - z = 0$ and $\overline{v}$ is perpendicular to this plane.

Solution : The vector $\overline{v}$ is the vector projection of $\overline{w}$ onto a vector $\overline{n}$ that is normal to the plane – we can take $\overline{n} = \bar{i} + 2\bar{j} - \bar{k}$. It follows that

$$\overline{v} = \frac{\overline{w}\cdot\overline{n}}{|\overline{n}|^2}\,\overline{n} = \frac{(3 - 10 - \sqrt{3})}{1+4+1}\,\overline{n} = \frac{-7-\sqrt{3}}{6}\ \left(\bar{i} + 2\bar{j} - \bar{k}\right).$$

We therefore have

$$\overline{u} = \overline{w} - \overline{v} = \left(\frac{25+\sqrt{3}}{6}\right)\bar{i} + \left(\frac{-8+\sqrt{3}}{3}\right)\bar{j} + \left(\frac{-7+5\sqrt{3}}{6}\right)\bar{k}.$$

5. Find the equation of the plane that contains the line $x = t, y = 3t, z = 0$ and the point $(2, -1, -1)$.

Solution : The plane must be parallel to the vector $\bar{i} + 3\bar{j}$. Since $(2, -1, -1)$ and $(0, 0, 0)$ both lie on the line, the plane must also be parallel to the vector $2\bar{i} - \bar{j} - \bar{k}$, and so a normal vector to our plane is

$$\overline{n} = \left(\bar{i} + 3\bar{j}\right) \times \left(2\bar{i} - \bar{j} - \bar{k}\right) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 3 & 0 \\ 2 & -1 & -1 \end{vmatrix} = -3\bar{i} + \bar{j} - 7\bar{k}.$$

An equation for our plane is thus (using the point $(0, 0, 0)$)

$$-3x + y - 7z = 0.$$

6. If $\overline{u}$ and $\overline{v}$ are unit vectors (i.e. vectors of magnitude 1) in $\mathbb{R}^3$, under what conditions would $\overline{u} \times \overline{v}$ also be a unit vector in $\mathbb{R}^3$? Justify your answer.

Solution : From the fact that

$$|\overline{u} \times \overline{v}| = |\overline{u}|\,|\overline{v}|\,\sin\theta,$$

where θ is the angle between $\overline{u}$ and $\overline{v}$, if we assume that $|\overline{u}| = |\overline{v}| = 1$, in order to have $|\overline{u} \times \overline{v}| = 1$, it is necessary and sufficient that $\sin\theta = 1$, i.e. that the vectors $\overline{u}$ and $\overline{v}$ are perpendicular.