Mathematics 100
Learning Objectives
Session 2018W Term 1 (September–December 2018)

Course Prerequisites

As a prerequisite to this course, students are required to have a reasonable mastery of precalculus mathematics (e.g., B.C. Principles of Mathematics 10–12), as this material will be required throughout the course. This includes being able to

- evaluate, manipulate, and simplify expressions containing basic mathematical functions, such as polynomial, radical, rational, trigonometric, exponential, logarithmic, absolute-valued, composite, piecewise functions (in particular, in expressions containing powers, exponentials and logarithms, perform algebraic manipulations by applying specific properties of such functions);

- solve linear, quadratic, rational, radical, trigonometric, exponential, logarithmic, and absolute-valued equations;

- relate the solutions to the above equations to intersections of graphs;

- solve linear and quadratic inequalities;

- find the domain and range of the above functions;

- construct new functions by applying function composition, identify the various functions that make up a composite function;

- find intersections of graphs with lines, coordinate axes, and other graphs;

- write down the equation of a line given two points on the line or one point and the slope, find the slope of a line given its equation, relate the slopes of parallel and perpendicular lines;

- determine whether a point of given coordinates lies on a certain curve, find the distance between two points;

- apply Pythagoras theorem, write down trigonometric relationships involving the sides and angles of a right triangle, express proportional relations between similar triangles;

- compute the area and volume of basic shapes and solids;

- relate the equation of a quadratic function to the graph of a parabola, and vice versa, and be able to draw the graph of a parabola given the equation;

- read the value of a function from its graph;

- for a moving object, compute the average speed over a certain time interval, or use information about the average speed to find the distance travelled or the time elapsed.

Course-level Learning Goals

In this course students will learn the basic ideas, tools and techniques of Differential Calculus and will use them to solve problems from real-life applications. Specifically, students will learn

- to analyze the behaviour of basic mathematical functions (polynomial, radical, rational, trigonometric, inverse trigonometric, exponential, logarithmic, absolute-valued, composite, piecewise functions) both graphically and analytically;

- to perform differentiation operations and other basic algebraic operations on the above functions and carry out the computation fluently;

- to recognize when and explain why such operations are possible and/or required;

- to interpret results and determine if the solutions are reasonable.

In addition, students will apply the above skills and knowledge to translate a practical problem involving some real-life application into a mathematical problem and solve it by means of Calculus. The applications include science and engineering problems involving velocity and acceleration of moving objects, rates of change, exponential growth and decay, approximations of functions, curve sketching, optimization. In general, when solving a problem students will be able to

- after reading a problem, correctly state in their own words what the problem is asking in mathematical terms and what information is given that is needed in order to solve the problem;

- after restating the problem, identify which mathematical techniques and concepts are needed to find the solution;

- apply those techniques and concepts and correctly perform the necessary algebraic steps to obtain a solution;

- interpret results within the problem context and determine if they are reasonable.

Students will also learn how to construct simple proofs. They will learn to show that a given mathematical statement is either true or false by constructing a logical explanation (proof) using appropriate Calculus theorems and properties of functions. In particular, when applying a theorem, students will recognize the importance of satisfying its hypotheses and drawing logical conclusions.

Top-level Learning Goals

Here are the major learning outcomes of the course. Not all of these outcomes are of equal importance. The level of difficulty of problems to be solved is indicated by examples given below, and also by the Suggested Homework Problems from the textbook assigned throughout the term, the Workshop problems (MATH 180), and past exams.

1. Be able to relate graphical, numerical and algebraic approaches to identifying the **limit** of a function at a point or at infinity, and be able to compute limits exactly or determine that a limit does not exist, using algebraic manipulations, limit laws and other theorems. Examples:

   (a) Evaluate, if it exists, $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$.

   (b) Evaluate, if it exists, $\lim\limits_{x \to -\infty} \dfrac{1 - x - x^2}{2x^2 - 7}$.

   (c) Find the value of the constant a for which $\lim\limits_{x \to -2} \dfrac{x^2 + ax + 3}{x^2 + x - 2}$ exists.

   (d) Evaluate, if it exists $\lim\limits_{x \to 0+} (2 + x^4) \cos\left(\dfrac{\pi}{x}\right)$.

2. Describe graphically the idea of a **continuous** function, and be able to relate this graphical representation to the **definition of continuity**. Be able to show whether or not a given function is continuous by computing limits and using theorems. State the Intermediate Value Theorem and use it to construct simple proofs about a given mathematical statement, specifically, recognize when the hypotheses of the theorem are satisfied, apply the theorem accordingly and draw logical conclusions based on it.

   Examples:

   (a) Sketch the graph of a function that is continuous everywhere except at $x = 3$ and is continuous from the left at $x = 3$.

   (b) Use the definition of continuity to show that $f(x) = x^2 + \sqrt{7 - x}$ is continuous at 4.

   (c) Determine all values of the constant b, if any exist, such that

   $$f(x) = \begin{cases} \frac{x^2 + x - 6}{x - 2} & \text{if } x \neq 2 \\ b & \text{if } x = 2. \end{cases}$$

   is continuous everywhere.

   (d) Determine all values of the constants A, B and C, if any exist, so that

   $$f(x) = \begin{cases} Ax^2 + Bx + C & \text{if } -\infty < x \leq 0 \\ x^{3/2} \cos(1/x) & \text{if } 0 < x < \infty. \end{cases}$$

   is continuous everywhere.

   (e) Prove that $2x - 1 - \sin x$ has at least one real root.

3. Compute **derivatives** and **higher derivatives** of basic functions, by using formulas and **differentiation rules** such as the Product, Quotient and Chain Rules. For example:

(a) Find the derivative of $a/(x + ae^x)$, where a is a constant.

(b) Find the derivative of $e^{\cos(x^2)}$.

(c) If $f(x) = (\cos x)^{\sin x}$, find $f'(x)$.

(d) Find the fourth derivative $f^{(4)}(x)$, of $f(x) = x^3 \ln x$.

4. Compute derivatives and higher derivatives of functions, by using the **definition of the derivative** as a limit. Be able to determine when a function is differentiable at a point. For example:

(a) State the definition of the derivative, and use it to find the derivative of $f(x) = x|x|$. No credit is given for the use of differentiation rules. Find the second derivative of $f(x)$. Explicitly state, with justification, the point(s) at which $f''(x)$ does not exist, if any.

(b) Use the definition of the derivative to determine all values of the constants A, B and C if any exist, so that

$$ f(x) = \begin{cases} Ax^2 + Bx + C & \text{if } -\infty < x \le 0 \\ x^{3/2} \cos(1/x) & \text{if } 0 < x < \infty. \end{cases} $$

is differentiable at 0.

5. Use different interpretations of the derivative, such as the **slope of a tangent line** to a graph, the **(instantaneous) velocity** of a moving object, or more generally the **(instantaneous) rate of change** of some quantity with respect to another quantity, to solve geometrical or physics problems such as:

(a) Find the equation of the tangent line to the curve $y = 1 + x + \sin x$ at the point $(0, 1)$.

(b) Find a point at which the tangent line to the curve $y = x\sqrt{3 - x}$ is vertical.

(c) The displacement (in metres) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds. Find the average velocity over the time interval $[3, 4]$, the instantaneous velocity when $t = 4$, and the instantaneous acceleration when $t = 4$.

(d) The mass of the part of a metal rod that lies between its left end and a point x metres to the right is $3x^2$ kg. Find the linear density when x is 2 m.

(e) Find an equation of a line that is tangent to both the curves $y = x^2$ and $y = x^2 - 2x + 2$ (at different points).

(f) A car is travelling at night along a highway shaped like a parabola with its vertex at the origin. The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue?

6. Determine if a function is **one-to-one** or not. If a function is one-to-one, find an explicit formula for its **inverse function**, and know the relationship between the graphs of a one-to-one function and its inverse. As specific examples of inverse functions, know the relationships between **logarithmic functions** and exponential functions, and between **inverse trigonometric functions** and trigonometric functions. Examples:

   (a) Find a formula for the inverse of the function $f(x) = x^2 + 5x + 4$, $-\frac{5}{2} \le x < \infty$.

   (b) Find a formula for the inverse of the function $f(x) = x^2 + 5x + 4$, $-\infty < x \le -\frac{5}{2}$.

   (c) Find the exact value of $\sin^{-1}(\sin(7\pi/3))$. Note: another notation for $\sin^{-1} x$ is $\arcsin x$.

   (d) Simplify $\cos(\sin^{-1} x)$ and $\sec(\tan^{-1} x)$. Note: another notation for $\tan^{-1} x$ is $\arctan x$.

7. Compute derivatives **implicitly**. Compute derivatives involving inverse trigonometric functions. Examples:

   (a) If $x^2 y^2 + x \sin y = 4$, find dy/dx.

   (b) For the curve defined by the equation $\sqrt{xy} = x^2 y - 2$, find the slope of the tangent line at the point $(1, 4)$.

   (c) Find the derivative of $\tan^{-1}(x^2)$. Note: another notation for $\tan^{-1} x$ is $\arctan x$.

8. Write down the **differential equation** that corresponds to **exponential growth or decay**, and its **solution**, explain why it is a solution. In problems involving applications of exponential growth and decay, recognize if it is appropriate to use the above equation, when it can be used directly, and when it can be used after performing a suitable substitution, then use the appropriate form for the solution to solve such problems. For example:

   (a) Two cities, Growth and Decay, have populations that are respectively increasing and decreasing at (different) rates that are proportional to the respective current population. Growth's population is now 3 million and was 2 million ten years ago. Decay's population is now 5 million and was 7 million ten years ago.

      i. How many years from now will Growth's population equal 4 million?
      ii. How many years from now will Growth and Decay have the same population?

   (b) When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a room that has a temperature of 20°C, the drink's temperature has increased to 10°C.

      i. Write down the differential equation satisfied by the temperature of the drink, assuming it satisfies Newton's Law of Cooling.
      ii. What is the temperature of the drink after *another* 25 minutes has elapsed?

9. Using the intepretation of the derivative as a rate of change and the technique of implicit differentiation, solve problems involving **related rates**, that is, problems involving rates of change of two or more quantities that depend on one other. Examples:

   (a) A baseball diamond is a square with side length 30 m. A batter hits the ball and runs toward first base with a speed of 8 m/s. When the batter is halfway to first base, at what rate is his distance from second base changing?

   (b) A trough is 10 m long and its ends have the shape of (inverted) equilateral triangles that are 2 m across. If the trough is being filled with water at a rate of 12 m^3/min, how fast is the water level rising when the water is 60 cm deep?

10. Explain what it means to approximate a function locally using a linearization, and how **linear approximations** arise from the geometrical interpretation of the derivative, using both graphical and analytical approaches. Relate the derivative and linear approximations to **differentials**. Compute linear approximations to functions at given points, and use them to estimate the value of the function in the vicinity of the point, for example:

   (a) Use a linear approximation to estimate $(1.999)^3$. Write your answer in the form $n/1000$, where n is an integer.

   (b) Using a suitable linear approximation, estimate $(8.06)^{2/3}$. Give your answer as a fraction in which both the numerator and denominator are integers.

11. Explain how **Taylor** (or **Maclaurin**) **polynomials** arise, and their connection to linear approximations. Compute the Taylor (or Maclaurin) polynomial of a certain degree at a given point for a given function. Estimate the size of the **error** in a Taylor polynomial using **Taylor's Formula with Remainder (The Lagrange Remainder Formula)**. Examples:

   (a) Find the 4th degree Maclaurin polynomial of $\cos x$.

   (b) Find the 5th degree Taylor polynomial of $\sin x$ at $x = \pi/2$.

   (c) If linear approximation is used to estimate $(1.999)^3$, find an upper bound for the absolute value of the error. Is the exact value greater than or less than the linear approximation? Explain in terms of Taylor's Formula with Remainder (the Lagrange Remainder Formula).

   (d) Determine what degree of Maclaurin polynomial would guarantee that the approximation of $\cos(0.25)$ using the Maclaurin polynomial would be within 0.001 of the exact value.

   (e) A function $f(x)$ has third derivative equal to $10/(1-x)$. The second-degree Taylor polynomial $T_2(x)$ at 0 is used to approximate $f(0.1)$. Find an upper bound for the error using Taylor's Formula, i.e. an upper bound for $|f(0.1) - T_2(0.1)|$.

12. Use derivatives to help find **maximum** and **minimum values** of functions, if they exist. Solve **optimization problems** by interpreting the idea of optimization as the procedure used to make a system or a design as effective or functional as possible (under given constraints), and translating it into a mathematical procedure involving finding the maximum/minimum of a function, then interpreting the results correctly. For example:

(a) Find the absolute maximum value of $f(x) = x/(x^2 + 1)$ on the interval $[0, 2]$.

(b) The function $f(x) = b/(x^2 + ax + 2)$ has a local maximum at $x = 1$, and the local maximum value $f(1)$ equals 2. Find the values of the constants a and b.

(c) ABC is a right triangle, with right angle at the point B and sides AB and BC both having fixed length L. A man wants to to from point C to point A by first walking to some point D between B and C and then walking directly in a straight line to point A. He can walk with velocity v_1 from C to D and velocity v_2 from D to A. If $v_1 = 2v_2$, find the distance of D from B that minimizes the time the man spends getting from C to A.

(d) A poster is to have a total area of 180 cm^2 with one-centimetre margins at the bottom and sides, and a two-centimetre margin at the top. What dimensions will give the largest printed area?

13. State the **Mean Value Theorem** and its corollaries, use it to construct simple proofs about a given mathematical statement, specifically, recognize when the hypotheses of the theorem are satisfied, apply the theorem accordingly and draw logical conclusions based on it:

(a) Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

(b) Prove that $2x - 1 - \sin x$ has exactly one real root.

(c) Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.

(d) Prove that the identity $2 \arcsin x = \arccos(1 - 2x^2)$ is valid for all $0 \leq x \leq 1$.

14. Explain how **derivatives affect the shape of a graph**. Use analytical and graphical approaches to explain what it means for a function to be increasing, decreasing, or constant, as well as concave upwards or downwards on an interval, and how it relates to the first and second derivative; for a given function, find the intervals where the function has such properties. Identify and locate, if they exist, any local maximum, minimum, and inflection point of a function. Recognize when a function has horizontal, vertical, or slant asymptotes, and find the equation of such lines when they exist. Sketch carefully by hand the graphs of given functions.

(a) Let $f(x) = x\sqrt{3 - x}$.

    i. Find the domain of f.

ii. Determine the x-coordinates of the local maxima and minima (if any) and intervals where $f(x)$ is increasing or decreasing.

iii. Determine intervals where $f(x)$ is concave upwards or downwards, and the x-coordinates of inflection points (if any).

iv. There is a point at which the tangent line to the curve $y = f(x)$ is vertical. Find this point.

v. The graph of $y = f(x)$ has no asymptotes. However, there is a real number a for which $\lim\limits_{x \to -\infty} f(x)/|x|^a = -1$. Find the value of a.

vi. Sketch the graph of $y = f(x)$, showing the features above and giving the (x, y) coordinates for all points occurring above and also all x-intercepts.

(b) The function $f(x)$ is defined by

$$f(x) = \begin{cases} e^x & \text{if } -\infty < x < 0 \\ \frac{x^2+3}{3(x+1)} & \text{if } 0 \le x < \infty. \end{cases}$$

i. Explain why f is continuous everywhere.

ii. Determine the x-coordinates of local maxima and minima (if any) and intervals where f is increasing or decreasing.

iii. Determine intervals (if any) where $y = f(x)$ is concave upwards or downwards.

iv. Determine the equations of any asymptotes (horizontal, vertical, or slant).

v. Sketch the graph of $y = f(x)$, showing the features above and the (x, y) coordinates of all points of interest above.

15. Explain what **antiderivatives** are, find the general form of the antiderivative of basic functions, and compute antiderivatives to solve simple initial value problems such as:

(a) Given that $f''(x) = 24x^2 + 6x + 10$, $f(1) = 10$, and $f'(1) = 20$, find $f(x)$.

(b) A particle moves in a straight line so that its velocity at time t seconds is $v(t) = t^3$ metres per second. If its position at time t seconds is $s(t)$ metres and $s(1) = 2$ metres, find $s(2)$.