1. Let $\zeta$ be a primitive $p$-th root of unity, where $p$ is an odd prime number. Show that $\mathbb{Z}[\zeta]^* = (\zeta)\mathbb{Z}[\zeta + \zeta^{-1}]^*$. Show that

$$\mathbb{Z}[\zeta]^* = \{\pm \zeta^k (1 + \zeta)^n : 0 \leq k < 5, n \in \mathbb{Z}\}$$

if $p = 5$. [It is believed that $(\zeta)$ denotes the cyclic group of order $p$ generated by $\zeta$. You may assume without proof the fact that the ring of integers of $\mathbb{Q}(\zeta)$ is $\mathbb{Z}[\zeta]$.]

2. Let $\zeta$ be a primitive $m$-th root of unity, $m \geq 3$. Show that the numbers $\frac{1 - \zeta^k}{1 - \zeta}$ for $(k, m) = 1$ are units in the ring of integers of the field $\mathbb{Q}(\zeta)$. The subgroup of the group of units they generate is called the group of cyclotomic units.

3. [rewritten slightly from textbook to lessen ambiguity] If $K/L$ is an extension of number fields and $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathcal{O}_K$, then one has $\mathfrak{a} = \mathfrak{a}\mathcal{O}_L \cap \mathcal{O}_K$, and $\mathfrak{a}|\mathfrak{b} \iff \mathfrak{a}\mathcal{O}_L | \mathfrak{b}\mathcal{O}_L$. 