All questions except come from section 2.6 of the course text.

1. Explain the intermediate value theorem using words and pictures. (Exercise 8.)

**Solution:** Let \( f(x) \) be a continuous function on the closed interval \([a, b]\). Call \( f(a) \) and \( f(b) \) its *endpoint values* of \( f \), and call a real number \( L \) an *intermediate value* of \( f \) if it satisfies

\[
f(a) \leq L \leq f(b) \quad \text{OR} \quad f(a) \geq L \geq f(b).
\]

The Intermediate Value Theorem simply states that \( f(x) \) attains all of its intermediate values, somewhere on this the closed interval. Equivalently: given any intermediate value \( L \), there exists some point \( c \in [a, b] \) satisfying \( f(c) = L \).

An analogy: if \( a < b \) then we intuitively know that, to get from \( x = a \) to \( x = b \), we have to pass through all the numbers \( c \) satisfying \( a \leq c \leq b \). The Intermediate Value Theorem says similarly that, in order to get from \( f(a) \) to \( f(b) \), my function has to take all the numbers \( L \) satisfying \( f(a) \leq L \leq f(b) \) (or possibly \( f(a) \geq L \geq f(b) \)).

Below is a graph illustrating the Intermediate Value Theorem.

For each of three intermediate values \( L_1, L_2, L_3 \), we have indicated by the dashed red lines the \( x \)-values which attain the desired intermediate value. That is,

\[
L_1 = f(x_1) = f(x_2) = f(x_3),
L_2 = f(y_1) = f(y_2) = f(y_3),
L_3 = f(z).
\]

The theorem itself does *not* tell us where these values are; we had to find them by considering the graph of the function.

2. Determine the points at which the following function \( f \) has discontinuities. At each point of
Solution: The continuity checklist states that a function \( f(x) \) is continuous at a point \( x = a \) if and only if the following three conditions are satisfied:

1. The domain of \( f(x) \) includes \( a \);
2. \( \lim_{x \to a} f(x) \) exists; and
3. \( \lim_{x \to a} f(x) = f(a) \).

The domain of our function \( f \) seems to consist of all real number from 0 to 4, apart from \( x = 3 \). In between each pair of consecutive integers, the graph of \( f \) consists of a nice, connected curve, so we will assume that it is continuous on these intervals. This leaves only the points \( x = 0, 1, 2, 3, \) and 4 to investigate.

First of all, we can see that

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2,
\]

hence \( \lim_{x \to 1} f(x) = 2 \) also. However, \( f(1) = 3 \), which is not the limiting value of the function, so condition 3. is violated, and \( f \) fails to be continuous at \( x = 1 \).

Second, we observe that

\[
\lim_{x \to 2^-} f(x) = 1 \neq 2 = \lim_{x \to 2^+} f(x),
\]

and so \( \lim_{x \to 2} f(x) \) fails to exist. Thus condition 2. (and, vacuously, condition 3.) are violated, and the function is not continuous at \( x = 2 \).

Third, we see that the function is not defined at \( x = 3 \), and so condition 1. is violated; we don’t even need to consider the other conditions to know that the function fails to be continuous at \( x = 3 \).
Finally, it is clear that \( \lim_{x \to 0^+} f(x) = 1 = f(0) \) and \( \lim_{x \to 4^-} f(x) = 2 = f(4) \). The expressions

\[
\lim_{x \to 0^-} f(x) \quad \text{and} \quad \lim_{x \to 4^+} f(x)
\]

are not well-defined, because they ask us to take values of \( f(x) \) as \( x \) approaches 0 (respectively, 4) at points which are outside the domain of our function. It is as ill-formed a concept as “the colour of Saturday.”

It makes sense in this case to call \( f(x) \) continuous at both endpoints \( x = 0 \) and \( x = 4 \), because it is right-continuous at the left endpoint and left-continuous at the right endpoint.

3. Let

\[
f(x) = \begin{cases} 
\frac{x^2 - 4x + 3}{x - 3} & \text{if } x \neq 3, \\
2 & \text{if } x = 3.
\end{cases}
\]

Determine whether or not \( f(x) \) is continuous at \( a = 3 \). Use the continuity checklist to justify your answer. (Exercise 18.)

**Solution:** We recall the continuity checklist from the solution to problem 2. Condition 1. is obviously satisfied. To check the remaining conditions, we observe that, whenever \( x \) is a real number not equal to 3, then \( x - 3 \neq 0 \), and therefore \( x - 3 \) is a quantity we can divide by. We have

\[
\frac{x^2 - 4x + 3}{x - 3} = \frac{(x - 3)(x - 1)}{x - 3},
\]

which clearly equals \( x - 1 \) whenever \( x \neq 3 \). In particular, when we take the limit as \( x \) approaches 3, we approach along values that do not equal 3. Therefore

\[
\lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x - 1)}{x - 3} = \lim_{x \to 3} x - 1,
\]

and because the function \( x - 1 \) is continuous (it is a polynomial), we know that its limit as \( x \) approaches 3 is its value at \( x = 3 \), namely \( 3 - 1 = 2 \), and therefore \( \lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3} = 2 \).

Because this equals the value of the function at \( x = 3 \), we see that all conditions of the checklist are satisfied, and therefore that the function is continuous at this point.

4. Determine the interval(s) on which the function \( f(x) = \frac{x^5 + 6x + 17}{x^2 - 9} \) is continuous (Hint: use theorem 2.10 from the textbook.) (Exercise 23.)

**Solution:** Theorem 2.10 tells us two things: one, that polynomials are continuous everywhere, and two, that functions of the form \( \frac{p(x)}{q(x)} \) (where \( p(x) \) and \( q(x) \) are themselves polynomials) are continuous whenever \( q(x) \neq 0 \).

The function \( f(x) = \frac{x^5 + 6x + 17}{x^2 - 9} \) is just such a function, and so by the theorem, it will be continuous whenever \( x^2 - 9 \neq 0 \), which is clearly when \( x = \pm 3 \), and nowhere else. It
follows that $f(x)$ is continuous everywhere apart from $x = \pm 3$, and so the largest intervals
on which it is continuous are

$$(-\infty, -3), \quad (-3, 3), \text{ and } (3, \infty)$$

(note that no interval contains 3 or $-3$).

Digging a bit deeper, we see that the numerator takes the value 278 when $x = 3$, and takes
the value $-244$ at $x = -3$. This means that, very close to $x = 3$ (respectively, $x = -3$),
the numerator of $f(x)$ will be positive (respectively, negative) and near 278 (respectively,
$-244$), and the denominator will be close to zero. So as we approach $x = \pm 3$, the value of
$f(x)$ will become arbitrarily large and positive (respectively, negative), and so no limit can
exist. Unlike the situation in some earlier questions, there is no way to “fix” this continuity
problem by redefining the value of $f(x)$ at the problem points, as happened in problem 3.

5. Let

$$f(x) = \begin{cases} 
2x & \text{if } x < 1, \\
x^2 + 3x & \text{if } x \geq 1.
\end{cases}$$

a. Use the continuity checklist to show that $f$ is not continuous at 1.

b. Is $f$ continuous from the left or from the right at 1?

c. State the interval(s) of continuity.

(Exercise 39.)

Solution:

a. Because when we take a left-hand limit, we only approach along values that are less
than the limiting point, it follows that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x.$$

Because $2x$ is a polynomial, it is continuous (by theorem 2.10), and because it is con-
tinuous, the limit of $f(x)$ as $x$ approaches $a$ is just $f(a)$. Thus

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2(1) = 2.$$

A similar argument gives us

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + 3x = (1)^2 + 3(1) = 4.$$

Because the left-hand limit and the right-hand limit do not coincide, the limit of $f(x)$ as
$x$ approaches 1 does not exist, and so condition 2. of the continuity checklist is violated.
It follows that $f(x)$ is not continuous at $x = 1$. 
b. From our work in part a. we see that 

\[ f(1) = 4 = \lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x). \]

By definition, a function is right-continuous at \( x = a \) if \( f(a) = \lim_{x \to a^+} f(x) \). Because this is true when \( a = 1 \), it follows that \( f(x) \) is right-continuous at \( x = 1 \). The corresponding statement is not true for left-continuity, so \( f(x) \) is not left-continuous at \( x = 1 \).

Alternatively, we know that \( f(x) \) cannot be left-continuous at \( x = 1 \), because then it would be continuous at that point (see problem 7a.), which is not the case by our work in part a.

c. Our work above shows that the only point of discontinuity of this function is \( x = 1 \). Therefore the set of all numbers \( x \) at which \( f(x) \) is continuous is \( \mathbb{R} \setminus \{1\} \).

However, per the textbook’s definition (see p. 103), the function \( f \) when restricted to the interval \([1, \infty)\) is continuous, because it is right-continuous at the left endpoint. It follows that the largest intervals on which \( f(x) \) is continuous are \((-\infty, 1)\) and \([1, \infty)\).

6. You are shopping for a $150,000, 30-year loan to buy a house. The monthly payment is 

\[ m(r) = \frac{150000(r/12)}{1 - (1 + r/12)^{-360}}, \]

where \( r \) is the annual interest rate. Suppose banks are currently offering interest rates between 6% and 8%.

a. Use the intermediate value theorem to show that there is a value of \( r \) in the interval \((0.06, 0.08)\) – i.e., an interest rate between 6% and 8% – that allows you to make monthly payments of $1000 per month.

b. Use a graph to illustrate your explanation to part a. Then determine the interest rate you need for monthly payments of $1000.

(Exercise 58.)

Solution:

a. Direct computation shows us 

\[ m(0.06) = \frac{150000(0.06/12)}{1 - (1 + 0.06/12)^{-360}} \approx 899.325, \]

whereas 

\[ m(0.08) = \frac{150000(0.08/12)}{1 - (1 + 0.08/12)^{-360}} \approx 1100.647. \]

We observe that 

\[ m(0.06) \leq 1000 \leq m(0.08), \]

and so if \( m(r) \) is continuous, then the Intermediate Value Theorem implies that there exists some \( c \in [0.06, 0.08] \) such that \( m(c) = 1000 \). Put another way, the Intermediate
Value Theorem implies that there exists an interest rate between 6% and 8% for which the monthly payments on the loan equal exactly $1000.

To show that \( m(r) \) is continuous, we write
\[
m(r) = \frac{150000(r/12)}{1 - (1 + r/12)^{-360}} = \frac{150000(r/12)}{1 - ((12 + r)/12)^{-360}} = \frac{150000(r/12)}{1 - (12/(12 + r))^{360}},
\]
to see that \( m(r) \) can be written as the quotient of polynomials, and use theorem 2.10.

b. The graph of \( m(r) \) on the interval \( r \in [0.06, 0.08] \) is very nearly a straight line. To approximate the interest rate which will guarantee a monthly payment of $1000, we use the graph to determine which interest rate corresponds to the desired payment, as illustrated in problem 1.

\[
\begin{align*}
\text{It appears that the desired interest rate is something like 7.01%.}
\end{align*}
\]

7. Determine whether or not the following statements are true, and provide an explanation or counterexample.

a. If a function is left-continuous and right-continuous at \( a \), then it is continuous at \( a \).

b. If a function is continuous at \( a \), then it is left-continuous and right-continuous at \( a \).

c. If \( a < b \) and \( f(a) \leq L \leq f(b) \), then there is some value of \( c \) in \( (a, b) \) for which \( f(c) = L \).

d. Suppose \( f \) is continuous on \( [a, b] \). Then there is a point \( c \) in \( (a, b) \) such that \( f(c) = \frac{f(a) + f(b)}{2} \).

\((Exercise 65.)\)

Solution:

a. True. If \( f(x) \) is left-continuous at \( a \), then by definition
\[
\lim_{x \to a^-} f(x) = f(a).
\]
Similarly if \( f(x) \) is right-continuous at \( a \), then
\[
\lim_{x \to a^+} f(x) = f(a).
\]
So, if \( f(x) \) is both left- and right-continuous at \( a \), then \( \lim_{x \to a} f(x) \) exists, and
\[
f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x).
\]
All conditions of the continuity checklist are satisfied, and we deduce that \( f(x) \) is continuous at \( x = a \).

b. True. If a function is continuous at \( a \), then according to the continuity checklist, the limit \( \lim_{x \to a} f(x) \) exists and equals both
\[
\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x),
\]
as well as \( f(a) \) itself. In particular, both left- and right-hand limits must equal \( f(a) \), and so
\[
\lim_{x \to a^-} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a^+} f(x) = f(a).
\]
By definition, \( f(x) \) is both left- and right-continuous at \( a \).

c. False. A counterexample is given by the function \( f(x) = 1 - x^2 \) on the interval \([-1, 1]\); here the only intermediate value is zero, because \( f(-1) = 0 = f(1) \). On the open interval \((-1, 1)\), one has \( f(x) > 0 \), and so there is no point strictly between \( a \) and \( b \) which satisfies \( f(x) = 0 \).

This does not violate the Intermediate Value Theorem because the statement of that theorem allows the possibility that \( c = a \) or \( c = b \), cases which are explicitly disallowed in our situation.

d. False. The same counterexample works in this situation, and it does not violate the Intermediate Value Theorem for the same reason as before. Note, however, that it is always the case that \( \frac{f(a) + f(b)}{2} \) is an intermediate value of \( f(x) \) on \([a, b]\).

8. Determine the value of the constant \( a \) for which the function
\[
f(x) = \begin{cases} 
\frac{x^2 + 3x + 2}{x+1} & \text{if } x \neq -1, \\
 a & \text{if } x = -1.
\end{cases}
\]
is continuous at \(-1\). (Exercise 84.)

**Solution:** For \( f(x) \) to be continuous at \( x = -1 \), we have to determine whether or not \( \lim_{x \to -1} f(x) \) exists, and if it does, to define \( f(-1) \) to be this limiting value. We have
\[
\frac{x^2 + 3x + 2}{x+1} = \frac{(x + 2)(x + 1)}{(x + 1)},
\]

and whenever \( x \neq -1 \), we know that \( x + 1 \neq 0 \) and so can be cancelled. In particular, we have

\[
\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{(x + 2)(x + 1)}{(x + 1)} = \lim_{x \to -1} x + 2 = (-1) + 2 = 1.
\]

It follows that, if we put \( a = 1 \), then the function \( f(x) \) is continuous at \(-1\).

9. Let

\[
f(x) = \begin{cases} 
  x^2 + x & \text{if } x < 1, \\
  a & \text{if } x = 1, \\
  3x + 5 & \text{if } x > 1. 
\end{cases}
\]

a. Determine the value of \( a \) for which \( g \) is continuous from the left at 1.
b. Determine the value of \( a \) for which \( g \) is continuous from the right at 1.
c. Is there a value of \( a \) for which \( g \) is continuous at 1? Explain why or why not. (Exercise 85.)

**Solution:** We proceed as in the previous solution.

a. We have

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + x = (1)^2 + (1) = 2,
\]

and so if we put \( a = 2 \), then \( f(1) = \lim_{x \to 1^-} f(x) \), and \( f(x) \) is left-continuous at 1.

b. We have

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3x + 5 = 3(1) + 5 = 8,
\]

and so if we put \( a = 8 \), then \( f(1) = \lim_{x \to 1^+} f(x) \), and \( f(x) \) is right-continuous at 1.

c. Because the left- and the right-hand limits at \( x = 1 \) do not coincide, the limit \( \lim_{x \to 1} f(x) \) does not exist, and so condition 2. on the continuity checklist will fail. It follows that there is no value of \( a \) for which \( f(x) \) is continuous at 1.