Very short answer question

1. [1 mark] True or false: the series $\sum_{n=1}^{\infty} \log(2n) - \log n$ is convergent.

Answer: False

Solution: This series fails the divergence test. The limit of the general term is 

$$\lim_{n \to \infty} \log(2n) - \log n = \lim_{n \to \infty} \log \left( \frac{2n}{n} \right) = \log 2 \neq 0.$$ 

It follows immediately by the divergence test that the series must diverge.

Marking scheme: 1 for the correct answer in the box
Short answer questions—you must show your work

2. 2 marks Solve the initial value problem

\[
\frac{dy}{dx} = xy^2, \quad y(0) = 1.
\]

Answer: \( y = \frac{2}{2 - x^2} \)

**Solution:** We begin by solving the differential equation \( y' = xy^2 \). One solution is \( y = 0 \); otherwise we can divide by \( y \). Separating the variables gives

\[
\frac{dy}{y^2} = x \, dx,
\]

and integrating both sides gives

\[-\frac{1}{y} = \frac{1}{2} x^2 + C.\]

Substituting \( x = 0, y = 1 \) gives

\[-\frac{1}{1} = \frac{1}{2} (0)^2 + C,\]

hence \( C = -1 \). Therefore the solution is given implicitly by

\[-\frac{1}{y} = \frac{1}{2} x^2 - 1,\]

and solving for \( y \) we have our final answer, namely

\[y = \frac{-1}{(1/2)x^2 - 1} = \frac{-2}{x^2 - 2} = \frac{2}{2 - x^2}.
\]

**Marking scheme:**

- 1 mark for solving the differential equation.
- 1 mark for correctly finding the constant and solving for \( y \) explicitly.
3. **2 marks** Evaluate the series \( \sum_{n=1}^{\infty} \frac{(-5)^{1-3n}}{6^{1-n}} \), and simplify your answer completely.

**Solution:** The general term of our series is \( a_n = \frac{(-5)^{1-3n}}{6^{1-n}} \), which we rewrite:

\[
a_n = \frac{(-5)^{1-3n}}{6^{1-n}} = \frac{6^{n-1}}{(-5)^{3n-1}} = \frac{-5}{6} \cdot \frac{6^n}{(-5)^{3n}} = -\frac{5}{6} \cdot \left( \frac{6}{(-5)^3} \right)^n.
\]

It follows that this is a geometric series, with common ratio \( r = \frac{6}{(-5)^3} = -\frac{6}{125} \). Because this is less than one, we can use the formula

\[
\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}
\]

to deduce

\[
\sum_{n=1}^{\infty} r^n = \frac{1}{1 - r} - 1 = \frac{r}{1 - r},
\]

and so we have

\[
\sum_{n=1}^{\infty} \frac{(-5)^{1-3n}}{6^{1-n}} = -\frac{5}{6} \sum_{n=1}^{\infty} \left( \frac{6}{(-5)^3} \right)^n = -\frac{5}{6} \left( \frac{-6/125}{1 - (-6/125)} \right),
\]

and we have our answer. Simplifying gives

\[
\sum_{n=1}^{\infty} \frac{(-5)^{1-3n}}{6^{1-n}} = -\frac{5}{6} \cdot -\frac{6}{125} \cdot \frac{125}{131} = \frac{5}{131}.
\]

**Marking scheme:**

- 1 point for rewriting the series as a geometric series.
- 1 point for using the correct formula to evaluate it.
Long answer question—you must show your work

4. **5 marks** Determine whether the series \( \sum_{n=3}^{\infty} \frac{(-1)^n n}{n^2 + 5n + 6} \) converges absolutely, converges conditionally, or diverges.

**Solution:** We first check whether or not the series is absolutely convergent; equivalently, we ask whether or not the series \( \sum_{n=3}^{\infty} \frac{n}{n^2 + 5n + 6} \) is convergent. Let \( a_n = \frac{n}{n^2 + 5n + 6} \) and let \( b_n = \frac{1}{n} \); then both \( a_n \) and \( b_n \) are always nonzero, and we have

\[
\frac{a_n}{b_n} = \frac{n}{n^2 + 5n + 6} \cdot \frac{n}{1} = \frac{n^2}{n^2 + 5n + 6} \to 1.
\]

By the limit comparison test, it follows that \( \sum_{n=3}^{\infty} a_n \) converges if and only if \( \sum_{n=3}^{\infty} b_n \) converges.

Because \( \sum_{n=3}^{\infty} b_n = \sum_{n=3}^{\infty} \frac{1}{n} \) is divergent (by the \( p \)-test, or from example 86), we know that \( \sum_{n=3}^{\infty} \frac{n}{n^2 + 5n + 6} \) diverges.

Therefore we know that \( \sum_{n=3}^{\infty} \frac{(-1)^n n}{n^2 + 5n + 6} \) is either conditionally convergent, or divergent.

Because the series is alternating, we see if we can apply the alternating series test: clearly \( \lim_{n \to \infty} \frac{(-1)^n n}{n^2 + 5n + 6} = 0 \), and we need only check that the sequence \( \frac{n}{n^2 + 5n + 6} \) is decreasing. One has

\[
a_n > a_{n+1} \iff \frac{n}{n^2 + 5n + 6} > \frac{n + 1}{(n + 1)^2 + 5(n + 1) + 6} \iff n((n + 1)^2 + 5(n + 1) + 6) > (n + 1)(n^2 + 5n + 6).
\]

We expand the polynomials on both sides of the inequality:

\[
n(n^2 + 2n + 1 + 5n + 5 + 6) > n(n^2 + 5n + 6) + (n^2 + 5n + 6) \iff n^3 + 7n^2 + 12n > n^3 + 6n^2 + 11n + 6,
\]

if and only if \( n^2 + n > 6 \). Because \( n^2 + n \) is an increasing function and \( 3^2 + 3 = 12 > 6 \), we see that this inequality is true for all \( n \geq 3 \), and so the general term of the series is indeed decreasing in absolute value.

Alternatively, we could have differentiated the function \( f(x) = \frac{x}{x^2 + 5x + 6} \) to obtain \( f'(x) = \frac{6-x^2}{(x^2 + 5x + 6)^2} \), which is negative whenever \( x > \sqrt{6} \approx 2.449 \); in particular, this means that \( f(x) \) (and therefore \( a_n \)) is decreasing for all integers \( \geq 3 \).

In either case, it follows by the alternating series test that \( \sum_{n=3}^{\infty} \frac{(-1)^n n}{n^2 + 5n + 6} \) converges, and we conclude that \( \sum_{n=3}^{\infty} \frac{(-1)^n n}{n^2 + 5n + 6} \) is conditionally convergent.
Marking scheme:

- 2 marks for showing the series diverges absolutely (using the comparison or limit comparison test).
- 2 marks for showing the series converges (using the alternating series test).
- 1 marks for stating “conditionally convergent” (i.e. using the correct terminology).

You may be deducted a mark if you do not check the premises of your convergence tests.