Very short answer question

1. [1 mark] Let \( f(x) \) be a positive, decreasing, continuous function on the interval \([a, b]\). We approximate the value of \( A = \int_a^b f(x) \, dx \) by taking a left and a right Riemann sum, each with \( n \) evenly spaced subintervals; call these approximations \( L_n \) and \( R_n \), respectively. List in increasing order the quantities \( L_n \), \( R_n \), and \( A \).

**Answer:** \( R_n \leq A \leq L_n \)

**Solution:** Let \( a = x_0 < x_1 < \cdots < x_n = b \) be the endpoints of our subintervals. If \( f(x) \) is positive and decreasing on \([a, b]\), then we have the chain of inequalities

\[
f(x_0) > f(x_1) > \cdots > f(x_n) > 0.
\]

So, on each subinterval \( x \in [x_{i-1}, x_i] \), we have \( f(x_{i-1}) \geq f(x_i) \), and hence

\[
f(x_{i-1}) \Delta x = \int_{x_{i-1}}^{x_i} f(x) \, dx \geq \int_{x_{i-1}}^{x_i} f(x) \, dx \geq \int_{x_{i-1}}^{x_i} f(x_i) \, dx = f(x_i) \Delta x.
\]

Taking the sum from \( i = 1 \) to \( n \), we have

\[
L_n = \sum_{i=1}^{n} f(x_{i-1}) \Delta x \geq \int_a^b f(x) \, dx \geq \sum_{i=1}^{n} f(x_i) \Delta x = R_n,
\]

and we’re done.

More intuitively: the rectangles making up the left Riemann sum will cover more than the area under the graph, and those of the right Riemann sum will cover less than the full area.

**Marking scheme:** 1 for the correct answer in the box
2. Write down the partial fraction decomposition of \( \frac{3x^2 - 2}{x^3 + x} \); you do not need to integrate the resulting expression.

**Answer:** \( -\frac{2}{x} + \frac{5x}{x^2 + 1} \)

**Solution:** The denominator \( x^3 + x \) factors as \( x(x^2 + 1) \), and is now the product of a non-repeated linear equation and an irreducible quadratic equation (because \( y = x^2 + 1 \) never crosses the \( x \)-axis). Therefore the partial fraction decomposition of \( \frac{3x^2 - 2}{x^3 + x} \) is

\[
\frac{3x^2 - 2}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1},
\]

for appropriate constants \( A, B, C \). We solve for them by rewriting the right-hand side:

\[
\frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{A(x^2 + 1) + (Bx + C)(x)}{x(x^2 + 1)} = \frac{(A + B)x^2 + C(x) + (A)}{x^3 + x}.
\]

We equate the polynomials \( 3x^2 - 2 \) and \( (A + B)x^2 + Cx + A \), giving the system of equations

\[
A + B = 3, \quad C = 0, \quad A = -2;
\]

the only constant that takes any work to solve is \( B = 5 \). Thus the partial fraction decomposition is

\[
\frac{3x^2 - 2}{x^3 + x} = -\frac{2}{x} + \frac{5x}{x^2 + 1},
\]

and we are done.

**Marking scheme:**

- 1 mark for using the correct form of the partial fraction decomposition.
- 1 mark for correctly finding the resulting constants.
3. **2 marks** The improper integral \( \int_2^\infty \frac{2}{4 + x^2} \, dx \) is convergent (you don’t have to show this). Find its value.

**Solution:** By definition,

\[
\int_2^\infty \frac{2}{4 + x^2} \, dx = \lim_{A \to \infty} \int_2^A \frac{2}{4 + x^2} \, dx.
\]

We solve this definite integral using the trigonometric substitution \( x = 2 \tan t \), so \( dx = 2 \sec^2 t \, dt \); this changes our limits of integration from \( x = 2, x = A \) to \( t = \arctan(1) = \frac{\pi}{4}, t = \arctan\left(\frac{A}{2}\right) \), and we have

\[
\int_2^\infty \frac{2}{4 + x^2} \, dx = \lim_{A \to \infty} \int_{\arctan(1)}^{\arctan\left(\frac{A}{2}\right)} \frac{2}{4 + 4 \tan^2 t} \, (2 \sec^2 t \, dt) = \lim_{A \to \infty} \int_{\frac{\pi}{4}}^{\arctan\left(\frac{A}{2}\right)} \frac{4 \sec^2 t}{4 \sec^2 t} \, dt;
\]
this last integrand is just 1, and we deduce

\[
\int_2^\infty \frac{2}{4 + x^2} \, dx = \lim_{A \to \infty} \left( \arctan\left(\frac{A}{2}\right) - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},
\]

and this is our final answer.

**Marking scheme:**

- 1 point for a correct trigonometric substitution.
- 1 point for correctly setting up and evaluating the limit.

You may be deducted a mark if you do not include any limits in your answer.
Long answer question—you must show your work

4. 5 marks Compute the antiderivative \( \int \frac{\sqrt{x^2 + 9}}{x^4} \, dx \); your final answer should be in terms of the variable \( x \).

Solution: We make the trigonometric substitution \( x = 3 \tan t \), so \( dx = 3 \sec^2 t \, dt \). Our integral becomes

\[
\int \frac{\sqrt{x^2 + 9}}{x^4} \, dx = \int \frac{\sqrt{9 \tan^2 t + 9}}{3^4 \tan^4 t} (3 \sec^2 t \, dt) = \int \frac{3^2 \sec^3 t}{3^4 \tan^4 t} \, dt.
\]

Writing this in terms of sine and cosine, we have

\[
\int \frac{\sqrt{x^2 + 9}}{x^4} \, dx = \frac{1}{9} \int \left( \frac{\cos t}{\sin t} \right)^3 \, dt = \frac{1}{9} \int \cos t \, \sin t^3 \, dt.
\]

We now make a \( u \)-substitution: \( u = \sin t, \, \, du = \cos t \, dt \), and we have

\[
\int \frac{\sqrt{x^2 + 9}}{x^4} \, dx = \frac{1}{9} \int \frac{du}{u^4} = \frac{1}{9} \cdot \frac{1}{u^3} + C = -\frac{1}{27 u^3} + C.
\]

In terms of \( t \), this is just \(-\frac{1}{27 \sin^3 t} + C = -\frac{\csc^3 t}{27} + C \), and we just have to figure out what \( \csc t \) is in terms of \( x \). Our substitution \( x = 3 \tan t \) can be read \( \tan t = \frac{x}{3} \), and we construct the corresponding right triangle:

```
     x
   \   / \n   \ /  \\  \sqrt{x^2 + 9}
   /    \\
 3 \  /     \\
   \    \\       t
   \   \\
   \ 
```

We see that \( \csc t = \frac{\sqrt{x^2 + 9}}{x} \), and we have our final answer:

\[
\int \frac{\sqrt{x^2 + 9}}{x^4} \, dx = -\frac{1}{27} \left( \frac{\sqrt{x^2 + 9}}{x} \right)^3 + C = -\frac{(x^2 + 9)^{3/2}}{27x^3} + C.
\]

Marking scheme:

- 1 mark for making a correct trigonometric substitution.
- 2 marks for dealing with the resulting integral (by any method).
- 2 marks for obtaining the final answer (in terms of \( x \)).

You may be deducted a mark if your answer includes something like \( \sin(\arctan(x/3)) \).