Very short answer question

1. [1 mark] The sequence \( \{a_n\}_{n=1}^{\infty} \), where \( a_n = n! \sin(n\pi) \), is convergent (you don’t need to prove this). Write down its limit.

Answer: 0

Solution: Because \( \sin(n\pi) = 0 \) for any integer \( n \), we have that \( a_n = n! \cdot 0 = 0 \), and so \( \{a_n\} \) is really the constant sequence \((0, 0, 0, \ldots)\). Clearly, the limit of this sequence is zero.

Marking scheme: 1 for any correct answer in the box

Short answer questions—you must show your work

2. [2 marks] Let \( \{a_n\} \) be the sequence defined recursively by the rule \( a_1 = 1, a_2 = 2, \) and \( a_n = \frac{a_{n-1}}{a_{n-2}} \) for \( n > 2 \).

(i) List the first five terms of the sequence.

(ii) State whether or not \( \{a_n\} \) is monotone, with justification (yes/no answers without justification will not be marked).

Answer: (i) 1, 2, 2, 1, \( \frac{1}{2} \)

(ii) not monotone

Solution: We solve the first part of the problem simply by plugging in values; that \( a_1 = 1, a_2 = 2 \) are given to us, and we need only calculate \( a_3, a_4, a_5 \), which we do using the formulas \( a_3 = \frac{a_2}{a_1}, a_4 = \frac{a_3}{a_2}, a_5 = \frac{a_4}{a_3} \).

To answer the second part of the problem, we need only observe that \( a_1 < a_2 \), yet \( a_3 > a_4 \), so this sequence can be neither increasing nor decreasing, and hence is not monotone.

Marking scheme:

- 1 mark for computing all five terms correctly.
- 1 mark for the answer of “not monotone,” with any reasonable justification.
3. **2 marks** Let \( \{a_n\} \) be the sequence defined by \( a_n = \frac{1}{(n+1)!} - \frac{1}{(n-1)!} \). Determine whether or not the sequence converges – you do not need to find its limit. (*Recall that \( 0! = 1 \) by definition.*)

**Solution:** We compute:

\[
\begin{align*}
a_n &= \frac{1}{(n+1)!} - \frac{1}{(n-1)!} = \frac{1}{(n+1)(n)(n-1)!} - \frac{1}{(n-1)!} = \frac{1}{(n-1)!} \left( \frac{1}{n(n+1)} - 1 \right).
\end{align*}
\]

So each term \( a_n \) is the product of \( b_n = \frac{1}{(n-1)!} \) and \( c_n = \frac{1}{n^2+n} - 1 \). The sequence \( b_n \) converges to zero, and the sequence \( c_n \) converges to \(-1\); it follows that \( a_n \), which is the product of \( b_n \) and \( c_n \), is itself convergent (to zero).

**Marking scheme:**
- 1 point for correctly stating that the sequence converges.
- 1 point for reasonable justification.

**Long answer question—you must show your work**

4. **5 marks** Consider the function \( f(x) = 2x + 1 \) on the interval \([0,1]\). Compute the right Riemann sum, using \( N \) subdivisions of equal length, that approximates the definite integral

\[
\int_0^1 f(x) \, dx,
\]

and take the limit as \( N \to \infty \) to determine the true value of the integral. (*Hint: recall the formula \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \).*

**Solution:** First of all, we divide the interval \([0,1]\) into \( N \) subintervals; as these subintervals must have equal length, we know this length must be \( \Delta x = \frac{1-0}{N} = \frac{1}{N} \). The endpoints of our intervals are therefore

\[
x_i = x_0 + i \Delta x = 0 + i \left( \frac{1}{N} \right) = \frac{i}{N},
\]

and because we are taking a *right* Riemann sum we put \( x_i^* = x_i \). Therefore, the Riemann sum we desire is

\[
\sum_{i=1}^{N} f(x_i^*) \Delta x = \sum_{i=1}^{N} \left( 2 \left( \frac{i}{N} \right) + 1 \right) \left( \frac{1}{N} \right) = \sum_{i=1}^{N} \left( \frac{2i}{N^2} + \frac{1}{N} \right).
\]

We will split this up into two sums like this:

\[
\sum_{i=1}^{N} \left( \frac{2i}{N^2} + \frac{1}{N} \right) = \sum_{i=1}^{N} \frac{2i}{N^2} + \sum_{i=1}^{N} \frac{1}{N}.
\]
The second sum is clearly just 1. To compute the first one, we use our formula:

\[
\sum_{i=1}^{N} \frac{2i}{N^2} = \frac{2}{N^2} \sum_{i=1}^{N} i = \frac{2}{N^2} \left( \frac{N(N+1)}{2} \right) = \frac{N^2 + N}{N^2}.
\]

Taking the limit as \( N \to \infty \), we have

\[
\int_{0}^{2} f(x) \, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i^*) \Delta x = \lim_{N \to \infty} \left( \frac{N^2 + N}{N^2} + 1 \right) = 2.
\]

**Marking scheme:**

- 1 mark for getting the correct endpoints (the \( x_i \)).
- 1 mark for the correct sample points for a *right* Riemann sum.
- 1 mark for the correct form of a Riemann sum.
- 1 mark for manipulating the sum properly.
- 1 mark for taking the limit.